INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

27 April 2022 (am)

Subject CM2 - Financial Engineering and Loss Reserving Core Principles

Paper A

Time allowed: Three hours and twenty minutes

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

If you encounter any issues during the examination please contact the Assessment Team on T. 0044 (0) 1865 268 873.

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1 A fund earns an annual rate of return i_t, with the rate of return in any year being independent of the rate in any other year. The distribution of $\log(1 + i_t)$ is normal with parameters μ and σ^2.

The mean of i_t is 5% and the standard deviation is 3%.

(i) Calculate μ and σ^2. [3]

(ii) Calculate the probability that the fund return for any year is between 1% and 3%. [3]

(iii) Comment on your answer to part (ii). [1]

A sum of £10,000 is invested into the fund.

(iv) Calculate the probability that the accumulated value of the fund at the end of 3 years is less than £11,000. [2]

[Total 9]

2 (i) Define the term 'loss ratio' as used in the Bornhuetter–Ferguson method for estimating outstanding claim amounts. [1]

The run-off triangle below shows cumulative claims incurred on a portfolio of insurance policies.

	Development year		
Accident year	*0*	*1*	*2*
2017	864	1,011	1,072
2018	798	915	
2019	820		

Annual premiums written for accident year 2019 were 1,520 and the ultimate loss ratio is assumed to be 92.5%. Claims can be assumed to be fully run off by the end of development year 2.

(ii) Calculate the total claims arising from accidents in 2019, using the Bornhuetter–Ferguson method. [5]

1 year later, an unexpected event has resulted in higher claims than expected. The run-off triangle is now as shown below.

	Development year		
Accident year	*0*	*1*	*2*
2018	798	915	1,320
2019	820	1,412	
2020	1,016		

(iii) Calculate the revised total claims arising from accidents in 2019, using the Bornhuetter–Ferguson method. [3]

(iv) Discuss the implications of your answer to part (iii) for the insurance company. [3]

[Total 12]

3 Consider a share, S_t, and a derivative on the share with a value at time t of $f(t, S_t)$.

(i) Define, in your own words, what is represented by each of the following Greeks for this derivative:

(a) Delta

(b) Gamma

(c) Vega.

[3]

Consider another share, A_t, which pays no dividends. The continuously compounded risk-free rate is r. Let K be the fair price at time 0 of a forward contract on A_t maturing at time T.

(ii) State the formula for K. [1]

Under the risk-neutral measure, Q, the share is expected to grow at the risk-free rate.

(iii) Demonstrate that the expected present value of the forward contract at time t $(0 \leq t \leq T)$ under the measure Q is $A_t - e^{-r(T-t)} K$. [2]

(iv) Calculate Delta, Gamma and Vega for the forward contract. [2]

(v) Comment on how the Greeks for the forward contract in part (iv) compare to the same Greeks for the underlying share. [2]

(vi) Discuss whether it would be appropriate to use a forward contract to Delta hedge a European call option on the share. [3]

[Total 13]

4 Suppose that under the unique equivalent measure martingale measure, Q, for a term structure model, the Stochastic Differential Equation satisfied by the instantaneous interest rate r is:

$$dr_t = \alpha(\mu - r_t)dt + \sigma dZ_t$$

where $\alpha > 0$, μ and σ are fixed parameters and under Q, Z is a standard Brownian Motion.

The process X is defined by:

$$X_t = r_t b(T - t) + \int_0^t r_s \, ds$$

where the function b is given by $b(s) = \frac{(1 - e^{-\alpha s})}{\alpha}$.

The function f is given by $f(x, t) = \exp(a(T - t) - x)$, where a is a differentiable function.

(i) Apply Ito's formula to $f(X_t, t)$. [6]

[Hint: You may use, without proof, the fact that $dX_t = A_t dt + B_t dZ_t$ where $A_t = \alpha\mu b(T - t)$, and $B_t = \sigma b(T - t)$.]

(ii) Find a differential equation that the function a must satisfy, in order for $f(X_t, t)$ to be a martingale. [2]

(iii) Determine an additional condition on a that is necessary for a bond with unit payoff at time T to have a price given by the formula:

$$B(t, T) = f(X_t, t)\exp\left(\int_0^t r_s ds\right)$$ [5]

[Total 13]

5 An insurance company holds a large amount of capital and wishes to distribute some to policyholders using one of two possible options.

Option A

A sum of £500 will be invested for each policyholder in a fund in which the expected annual effective rate of return is 3.5% and the standard deviation of annual returns is 2%. The annual rates of return are independent and $(1 + i_t)$ is log-normally distributed with parameters μ and σ^2, where i_t is the rate of return in year t. The policyholder will receive the accumulated investment at the end of 15 years.

Option B

A sum of £500 will be invested for each policyholder for 10 years at a fixed rate of return of 4% p.a. effective. After 10 years, the accumulated sum will be invested for a further 5 years at the prevailing 5-year spot rate. This spot rate follows the probability density function shown below:

Spot rate (% p.a.)	*Probability*
0.5	0.15
1.0	0.25
4.5	0.40
7.0	0.20

The policyholder will receive the accumulated investment at the end of the 15 years.

(i) Demonstrate that $\mu = 0.0342$ and $\sigma = 0.0193$. [4]

(ii) Calculate the expected value and standard deviation at the end of year 15 of:

(a) Option A.

(b) Option B.

[12]

(iii) Determine, for each of Options A and B, the probability that a policyholder's accumulated investment at the end of the 15 years will be less than £775. [5]

(iv) Compare the relative risk of the two options. [2]

[Total 23]

6 Consider a share with price S_0 at time $t = 0$. The share will pay out a dividend of X at time $t = 1$ and again at time $t = 2$. The continuously compounded risk-free rate is r per unit of time.

Assume that the dividend payments are reinvested at the risk-free rate.

(i) Demonstrate that the fair price of a forward contract on S_t maturing at time $T > 2$ is $K = (S_0 - I)e^{rT}$, where I is the present value of the two dividends. [4]

A share is worth \$100 at time $t = 0$. It will pay a dividend of \$5 at time $t = 1$ and again at time $t = 2$. The continuously compounded risk-free rate is 5% per unit of time.

(ii) Calculate the fair price of a forward contract on the share maturing at time $T = 3$. [2]

An investor takes a long position in the forward contract in part (ii) at time $t = 0$. Immediately afterwards, the proposed dividends on the underlying share are cancelled.

(iii) Discuss the implications of this for the investor. [2]

[Total 8]

7 An investor makes decisions based on the utility function $U(w) = w - 6w^2$, where w is the investor's wealth in millions of dollars (\$m).

(i) Demonstrate that the investor has both increasing absolute and relative risk aversion. [3]

The investor has \$50,000 to invest over a 1-year period and has no other wealth. They have three options:

A Invest in a risk-free account. There will be no change in the value of the investment over 1 year.

B Invest in an asset that will give a 60% return over 1 year with probability 0.2, a 20% return with probability 0.7 and a –40% return with probability 0.1.

C Invest in an asset that will give a 30% return with probability 0.5 and a 20% return with probability 0.5.

The investor makes no allowance for discounting when making investment decisions. The investor must invest the whole \$50,000 in a single option.

(ii) Determine which option the investor should choose to maximise their expected utility at the end of the year. [5]

(iii) Comment on why the investor could not use $U(w)$ to choose from the above options if their initial wealth was \$65,000. [2]

[Total 10]

8 Consider a company funded entirely by debt and equity. The total value of the company's assets is \$40 million. It has debt with a current outstanding amount of \$20 million, with a continuously compounded interest rate of 8% p.a. and maturity in 7 years. Interest is added to the outstanding debt to be paid at maturity.

The volatility of the company's total assets is 10% p.a. The continuously compounded risk-free rate of interest is 2% p.a.

(i) Calculate the outstanding value of the debt, with interest, at maturity. [1]

(ii) Calculate the current value of the company's equity using the Merton model. [5]

(iii) Calculate the implied probability that the company will have sufficient assets to repay the debt at maturity. [1]

(iv) Demonstrate that the implied value of the company's debt is \$29.7 million to the nearest \$0.1 million (where the total value of the company is the value of the equity plus the value of the debt). [1]

(v) Explain why your answers to parts (i) and (iv) are different. [2]

An analyst wishes to set up a two-state credit model for the company. The two states will be denoted 'solvent' and 'default' with a constant transition intensity, λ, from solvent to default.

(vi) Calculate the value of λ that gives the same probability of default at maturity of the debt as calculated in part (iii). [2]

[Total 12]

END OF PAPER