

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

17 April 2024 (am)

Subject CS2 – Risk Modelling and Survival Analysis Core Principles

Paper A

Time allowed: Three hours and twenty minutes

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

If you encounter any issues during the examination please contact the Assessment Team on T. 0044 (0) 1865 268 873.

1 Suppose that the size of an insurance claim, X , has the following density function:

$$f(x) = \frac{1}{3x\sqrt{2\pi}} e^{-z^2/2}$$

for $x > 0$ where $z = \frac{\ln(x)-2.6}{3}$.

The insurance coverage pays claims subject to a deductible of £2,000 per claim.

- (i) State the distribution of X and its parameters. [1]
- (ii) Calculate the expected claim amount paid by the insurer per claim assuming that the claims occur at the end of the year with inflation of 20% p.a. [10]
[Total 11]

2 The following table shows mortality data and graduated mortality rates of male and female members of a pension scheme:

<i>Gender</i>	<i>Age (x years)</i>	<i>Exposed-to-risk (years)</i>	<i>Observed deaths</i>	<i>Graduated rates</i>
Female	70	1,007	153	0.14227
Female	71	978	166	0.16530
Female	72	1,111	204	0.19205
Female	73	1,500	326	0.22313
Female	74	1,200	306	0.25924
Female	75	2,001	599	0.30119
Male	70	908	338	0.36788
Male	71	998	439	0.42742
Male	72	1,009	497	0.49659
Male	73	877	508	0.57695
Male	74	859	571	0.67032

The data was graduated using the following parametric formula:

$$\log(\mu_x) = \alpha + \beta I_F + \delta x$$

where I_F is an indicator variable taking the value 1 for female members and 0 for male members.

- (i) Calculate the values of α , β and δ , by considering the graduated rates at ages 70 and 71. [4]
- (ii) Comment on your results. [1]
- (iii) Perform a Chi-square test at a 5% significance level to assess the overall appropriateness of these graduated rates for both male and female members, stating your null and alternative hypotheses. [5]
- (iv) State two limitations of the Chi-square test in this context and in each case suggest an alternative test that would address that limitation. [2]
[Total 12]

3 An analyst is trying to fit the following time series model:

$$y_t = a_2 y_{t-2} + a_3 y_{t-3} + \varepsilon_t$$

- (i) Show that the Yule–Walker equations for lags 1 and 2 are $\rho_1 = a_2 \rho_1 + a_3 \rho_2$ and $\rho_2 = a_2 + a_3 \rho_1$. [2]
- (ii) Derive expressions for the autocorrelation function for lags 1 and 2. [2]
- (iii) Derive the values of the partial autocorrelation function, ϕ_k , for lags $k = 1, 2, 3, 4$ and 5 , if $a_2 = 0.3$ and $a_3 = -0.2$. [6]
- [Total 10]

4 An insurer has a portfolio of independent policies under which the number of claims follows a Poisson process with 300 claims expected per annum. Claim amounts are exponentially distributed with mean £3,000. Let S denote aggregate annual claims from the portfolio. A check is made for ruin only at the end of the year. The insurer includes a loading of 12% in the premiums for all policies. Expenses are ignored.

- (i) Estimate the initial capital required, U , using a Normal approximation to the distribution of S , in order that the probability of ruin at the end of the first year is 5%. [6]

The insurer is considering purchasing proportional reinsurance from a reinsurer that includes a loading, β , in its reinsurance premiums. The proportion of each claim to be retained by the direct insurer is 80%.

Let S_1 denote the aggregate annual claims paid by the direct insurer net of reinsurance and U_1 denote the required initial capital after purchasing the reinsurance. The insurer uses a Normal approximation to the distribution of S_1 .

- (ii) Calculate the maximum reinsurance loading β for which U_1 is less than U given the same 5% ruin probability. [9]
- [Total 15]

5 Apple trees growing in an orchard are classified by the farmer in one of three states:

- F healthy trees with fruit
- N unhealthy trees that give no fruit
- D dead trees.

Every summer the farmer examines each tree and records its state. Year-to-year changes in tree classification are modelled using a Markov chain with the following 1-year transition probabilities:

	<i>F</i>	<i>N</i>	<i>D</i>
F	0.96	0.03	0.01
N	0.24	0.73	0.03
D	0	0	1

- (i) Calculate the probability that a healthy tree will be rated unhealthy in 2 years. [2]
 - (ii) Calculate the percentage of healthy trees that are dead in 2 years. [2]
 - (iii) Calculate the probability that a healthy tree will never be rated unhealthy. [4]
 - (iv) Explain how this model could be adapted if the farmer wanted to take into account the age of each tree. [4]
- [Total 12]

6 X and Y are random variables with $X \sim N(0, 1)$ and $Y \sim N(0, 1)$. For the purposes of parts (i) to (iv), the dependence between X and Y is modelled using a Gumbel copula with a copula parameter equal to 1.5.

- (i) Calculate $P(X < -1.64, Y < -2.33)$. [3]
 - (ii) Calculate $P(X > 1.64, Y > 2.33)$. [3]
 - (iii) Calculate $P(-1.64 < X < 1.64, -2.33 < Y < 2.33)$. [3]
 - (iv) Comment briefly on the key properties of the Gumbel copula using your answers to parts (i) and (ii). [3]
- [Total 12]

7 Let X_t be the zero-mean time series process defined by:

$$X_t = aX_{t-1} + be_{t-1} + X_{t-1}e_{t-1} + e_t$$

where e_t is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables and a and b are positive constants with $a^2 + \sigma^2 < 1$.

(i) Demonstrate that $E(X_t e_t) = \sigma^2$. [3]

(ii) Demonstrate that $E(X_t e_{t-k}) = a^{k-1}((a+b)\sigma^2 + \sigma^4)$ for $k \geq 1$. [9]

By calculating further similar expectations it can be shown that the autocorrelation function, ρ , of X_t is of the form:

$$\rho_k = Aa^k + Bka^k$$

for $k \geq 1$, for suitable positive constants A and B .

Let Y_t be the zero-mean time series process defined by:

$$Y_t = aY_{t-1} + e_t$$

(iii) Explain the similarities and differences in shape between the autocorrelation functions of X_t and Y_t . [4]

[Total 16]

8 The following matrix is the generator matrix of a three-state continuous time Markov chain:

$$G = \begin{pmatrix} a & 2 & 1 \\ 2.5 & -3 & b \\ 0 & c & 0 \end{pmatrix}$$

(i) Calculate the values of a , b and c . [2]

(ii) Derive the probability that the chain will leave state 2 sometime before 0.5 time units, given that the chain is in state 2 at time 0. [4]

(iii) Derive the transition matrix of the corresponding Markov jump chain. [6]

[Total 12]

END OF PAPER