INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

19 September 2022 (am)

Subject CS2 – Risk Modelling and Survival Analysis
Core Principles

Paper A

Time allowed: Three hours and twenty minutes

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

If you encounter any issues during the examination please contact the Assessment Team on T. 0044 (0) 1865 268 873.

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1 A leading smartphone manufacturer has developed fast chargers for its mobile phones. Currently, the manufacturer is testing the life of batteries using such chargers. 'Battery life' is defined as 'the number of times the battery can be charged using such chargers'. It has conducted many tests in its laboratories and has found the following:

- 70% of the batteries survived 400 charges.
- 50% of the batteries survived 810 charges.

Calculate, with suitable workings, the number of batteries that would survive 1,000 charges if battery life follows a Weibull distribution. [7]

2 Consider the time series process, X_t, given by:

$$X_t = aX_{t-1} + \tfrac{1}{2}X_{t-2} + e_t + be_{t-1}$$

where e_t is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables.

Determine the values of the parameters a and b such that X_t is:

(i) stationary. [5]

(ii) invertible. [2]

(iii) $I(1)$. [2]

[Total 9]

3 Consider the following stochastic process:

$$X_t = \sum_{i=0}^{\infty} 0.5^i e_{t-i}$$

where e_j is a sequence of independent and identically distributed random variables with a mean of zero and variance σ^2, for $j = 0, \pm 1, \pm 2, \ldots$

(i) Determine whether X_t is stationary and satisfies the Markov property. [4]

(ii) Determine whether your conclusions from part (i) also apply to the process $Y_t = X_t - 0.3X_{t-1}$. [6]

[Total 10]

4 When considering rates of mortality, actuaries generally use the initial rate of mortality, q_x, whereas demographers often use the central rate of mortality, m_x. In order to estimate m_x for each age x, a demographer collects the following data:

Age at last birthday	*Observed number of deaths*	*Central exposed to risk in years*
60	34	4,267
61	36	4,037
62	42	4,269
63	47	4,299
64	46	3,929

(i) Explain the difference between the two definitions of rate of mortality, stating which rate has the higher value at a given age, x. [2]

(ii) Estimate m_x for $x = 60, 61, 62, 63$ and 64 years using the data above. [2]

(iii) A pension scheme starts paying pensions on a member's 65th birthday. Calculate the probability that the pension scheme member aged 60 exact will start receiving a pension, using the estimates in part (ii) above and carefully explaining all the steps and assumptions needed. [4]

[Total 8]

5 An insurance company has graduated the experience of one block of its life business using the following quadratic-Gompertz formula:

$$\mu_x = \exp(a_0 + a_1 x + a_2 x^2)$$

where a_0, a_1 and a_2 are constants and x is age in years.

The data used for this graduation exercise, together with the graduated rates, are shown in the table below. All data have been collated between 1 January 2018 and 31 December 2018 inclusive on an 'age nearest birthday' basis.

Age	*Exposed to risk (years)*	*Death counts*	*Graduated rates*
60	11,362	80	0.0069
61	11,086	85	0.0077
62	10,816	100	0.0085
63	10,530	105	0.0095
64	10,301	108	0.0105
65	10,233	117	0.0115
66	9,970	110	0.0126
67	9,708	131	0.0138
68	9,667	145	0.015
69	9,807	140	0.0163

(i) Perform a Chi-square test to assess the overall goodness of fit of this graduation, stating the null and alternative hypotheses. [5]

The company would like to extend the graduation to age 70. The death count and exposed to risk at age 70 are 150 deaths and 10,000 years, respectively.

Let $\hat{\mu}_{70}$ be the graduated rate at age 70, and let the null and alternative hypotheses be the same as in part (i), except that the age range is now from 60 to 70.

(ii) Determine, using the data in the table above, the condition that $\hat{\mu}_{70}$ would need to fulfil in order for there to be insufficient evidence, at the 5% significance level, to reject the null hypothesis under the cumulative deviations test. [6]

[Total 11]

6 Consider the zero-mean MA(1) process, X_t, given by:

$$X_t = e_t + be_{t-1}$$

where e_t is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables and $|b| \leq 1$.

A sample x_t, $t = 1, 2, \ldots, T$, is taken, which is believed to be from X_t, and it is assumed that $e_0 = 0$.

(i) Demonstrate that the log-likelihood function is given by:

$$L(b, \sigma^2) = \text{constant} - \tfrac{1}{2}\, T \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{T} \left(x_t + \sum_{i=1}^{t-1} (-b)^i x_{t-i} \right)^2 .$$

[6]

You are given that the maximum likelihood estimator of b, $\hat{b}$, obtained by setting the partial derivatives of L to zero satisfies, under appropriate assumptions, the following:

$$\frac{\hat{b}}{1 + \hat{b}^2} = r_1,$$

where r_1 is the sample autocorrelation of x_t at lag 1.

(ii) Determine $\hat{b}$, considering separately the cases:

$$-\tfrac{1}{2} \leq r_1 < 0,\ r_1 = 0 \text{ and } 0 < r_1 \leq \tfrac{1}{2}.$$

[4]

(iii) Explain what conclusion should be drawn if $|r_1| > \tfrac{1}{2}$. [3]

[Total 13]

7 An insurer writes three different classes of insurance business: X, Y and Z. The classes have the following total annual claims distributions:

$X \sim \text{Exp}(0.08)$
$Y \sim \text{Normal}(10, 2^2)$
$Z \sim \text{Normal}(20, 3^2)$

The insurer models the level of dependency between the classes' total claim amounts using a Clayton ($\alpha = 2$) copula.

(i) Calculate the probability, using the Clayton (2) copula, that both $X < 3$ and $Y < 8$. [3]

(ii) Calculate the probability, using the Clayton (2) copula, that all of the following occur: $X < 3$, $Y < 8$ and $Z < 20$. [3]

(iii) Calculate the probability, using the Clayton (2) copula, that both $X > 10$ and $Y > 12$. [3]

A student has noted that copulas are a useful modelling tool as 'they allow us to model different degrees of dependency'.

(iv) Comment on this statement. [3]

[Total 12]

8 A combined 1-year life and health insurance policy pays £10,000 on death within the year and £15,000 on diagnosis of critical illness within the same term. Once the critical illness benefit is paid, no further benefits are payable. On 1st January 2021, the insurance company had 887 policies in force for people aged 56 at last birthday, of which 849 were still active at the end of the year. During 2021, the insurance company paid £115,000 in claims to policyholders aged 56 at last birthday, of which £75,000 were for critical illness.

(i) Draw a diagram of a three-state model that could be used to evaluate premiums for this policy, fully labelling all states and transition intensities. [3]

(ii) Beginning with the likelihood function for the model from part (i) above, derive and then solve equations for the maximum likelihood estimates of the transition intensities, stating any assumptions you make. [5]

(iii) Estimate the expected cost of critical illness benefit for a new policyholder aged 56 exact. [3]

(iv) Explain why the insurance company should be cautious about using the estimate from part (iii) above in premium calculations. [4]

[Total 15]

9 An airline company is concerned about potential financial losses due to currency movements in various countries. An analyst has been hired to model daily currency movements in these economies. The objective is to understand the likelihood of extreme currency movements that may have major financial consequences for the airline.

The analyst has suggested fitting several types of standard statistical distributions to the entire datasets, using maximum likelihood estimation, including Normal, Gamma, Pareto and lognormal. Estimates can then be made regarding probabilities of extreme currency movements based on the chosen model.

(i) Discuss the problems with this approach. [4]

The analyst proceeds to use the Generalised Pareto Distribution (GPD) to assess only extreme daily losses. For this analysis, currency losses were treated as positive, and gains were treated as negative.

A threshold was set at daily losses of +2.2%, resulting in a data subset of 150 'extreme' values from the 3,000 original data points to be used in the extreme value analysis. The following results were obtained from the subsequent analysis:

- Beta = 0.009785.
- Gamma = 5.417 (shape).

(ii) Calculate G(0.02) using the above parameter values, where G is the distribution function of the GPD, and write down its meaning. [3]

(iii) Estimate the probability that the currency losses tomorrow exceed 5%, based on this GPD model. [3]

A Normal distribution was also fitted to the data. The best fitting Normal distribution was determined to be $N(0, 0.014^2)$. However, currency returns are generally considered to exhibit leptokurtic behaviour.

(iv) Calculate the probability that daily losses exceed 5% using this Normal distribution, and comment on the results. [3]

(v) Discuss further investigations the analyst should proceed with, including any problems with using extreme value theory and/or the GPD in this case. [2]

[Total 15]

END OF PAPER