A Semi-Markov Multiple State Model for Reverse Mortgage Terminations *

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January 24, 2011

Abstract

Reverse mortgages provide a way for seniors to release the equity that has been built up in their home to supplement their post-retirement income. The value of a reverse mortgage loan is heavily dependent on the maturity or termination date, which is uncertain. In this research, we model reverse mortgage terminations using a semi-Markov multiple state model, which incorporates three different modes of termination: death, entrance into a long-term care facility, and voluntary prepayment. We apply the proposed model specifically to develop the valuation formulas for roll-up mortgages in the U.K. and Home Equity Conversion Mortgages (HECMs) in the U.S.A. We examine the significance of each mode of termination by valuing the contracts allowing progressively for each mode. On the basis of our model and assumptions, we find that both health related terminations and voluntary (non-health related) terminations significantly impact the contract value. In addition we analyze the premium structure for US reverse mortgage insurance, and demonstrate that premiums appear to be too high for some borrowers, and substantial cross-subsidies may result.

*A dissertation presented to the Institute and Faculty of Actuaries in partial fulfilment of the requirements for the Specialist Applications Dissertation option (SA0) for the qualification of Fellow of the Institute and Faculty of Actuaries.

Acknowledgement

I owe my deepest gratitude to my supervisors, Mary R. Hardy and Johnny Siu-Hang Li, for their support, guidance and encouragement throughout the research.

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1 Introduction

A reverse mortgage is a loan available to seniors to convert home equity to cash or lifetime income. The elderly borrow money against the value of their home equity and retain full ownership of the home for the whole life of the loan. No repayments of interest or principal are required until the last survivor dies or leaves home (e.g., moves to a long-term care facility) permanently. At that time, the mortgaged home is sold and the proceeds from the sale are used to repay the loan. Usually, a reverse mortgage includes an embedded guarantee which ensures that the borrower will not have to pay back any more than the value of the mortgaged home if it is less than the amount owing on the loan. This guarantee is called the No-Negative-Equity-Guarantee (NNEG) in the UK and the non-recourse provision (NRP) in the US. We will use the NNEG acronym here to refer to the guarantee whether in the UK or US context.

From the lender's viewpoint, the inclusion of the NNEG is the same as writing a put option on the mortgaged home with the strike price being the outstanding balance of the loan when the loan is terminated. Termination occurs when the property is vacated, or on earlier prepayment. The payoff from the guarantee is determined by the interest rate on the loan, the house price appreciation rate, and the termination date. In this research, we specifically focus on the uncertainty associated with the termination date. This piece of uncertainty is critically important, because the longer the loan continues, the more likely that the outstanding balance will exceed the net house value. From an option valuation perspective, the strike price is increasing at the interest rate on the loan, which will be greater than the risk free rate. In this case, the value of the put option is an increasing function of the term. We therefore require a model for reverse mortgage terminations to value the contract, and to better understand the risks and impact of borrower behaviour.

Currently, multivariate regression models are used for reverse mortgage terminations in the US. The model was originally suggested by Chow et al. (2000), and was adapted by the US Department of Housing and Urban Development (2003) and Rodda et al. (2004). These models provide a good fit to the actual termination rates before age 90. However, because they are regression-based, they require substantial economic and non-economic information about the borrowers as input. Another problem of these models is that they assume the probability of loan termination remains level after age 90; this is counter-intuitive, and because there is significant longevity risk inherent in the guarantee, appears to be a significant weakness. Furthermore, these models do not make explicit allowance for moveouts, health or non-health related.

In this work, we use a multiple state model to model reverse mortgage terminations. Multiple state models are extensively used in analyzing longitudinal failure times, particularly in the modeling of dependent failures. They have also been used by actuaries in a wide range applications; see Dickson et al. (2009) for examples. Pertinent to this work, Rickayzen and Walsh (2002) use a multiple state model to project the number of people with disabilities in the UK over a period of 35 years. In this work we utilize and extend the semi-Markov model of Ji et al. (2010), which describes the dependence between the lifetimes of a husband and wife.

The model proposed in this research explicitly incorporates three different modes of termination: death, entrance into a long-term care facility, and voluntary prepayment. In addition, the event-triggered dependency between the lifetimes of a husband and wife is modeled. This feature is of practical importance, because a significcant proportion of reverse mortgages are issued to couples (around 40% in the US, according to HECM, (2009)). The model would also offer a more sophisticated approach to reverse mortgages purchased by widows/widowers.

The rest of this dissertation is organized as follows: in Section 2 we provide some background information regarding reverse mortgages in the UK and US, and describe the guarantees embedded in the reverse mortgages sold in these two countries. Section 3 first discusses different modes of reverse mortgage termination; it then describes the semi-Markov multiple state model which we use for modeling these modes of termination. Section 4 applies the model to roll-up mortgages sold in the UK, and examines the importance of each mode of termination to the value of the embedded guarantee. Section 5 applies the proposed model to HECMs sold in the US, and analyzes the adequacy of the guarantee premiums that are currently charged. Finally, Section 6 concludes the work.

2 Reverse Mortgages in the UK and US

2.1 Contract Design

Reverse mortgages are available in many countries, including the UK, the US, India, Australia, and Japan. In this work, we consider specifically roll-up mortgages in the UK, and Home Equity Conversion Mortgages (HECMs) in the US for illustrative purposes. Below we provide some background information about these two types of reverse mortgage.

In the UK, there are different ways for older home owners to release the equity that has been built up in their home. One way is to use a home reversion, which is not a mortgage but a sale with conditions. Under a home reversion contract, the homeowner sells all or part of his/her property to the provider for an agreed amount, but retains the right to live in the property rent-free until death. Another way is relying on a lifetime mortgage, which permits homeowners to borrow money against the value of their home equity and retain full ownership of the home for the whole life of the loan. Depending on how the loan is taken and repaid, lifetime mortgages are divided into finer classifications. In this research, we consider roll-up mortgages, also called fixed interest lifetime mortgages, which may be regarded as the most straightforward type of lifetime mortgages. Other types of lifetime mortgages include interest-only mortgages and drawdown mortgages. We refer interested readers to the Institute of Actuaries (2005a) for further details.

In a typical roll-up mortgage, the homeowners are advanced a lump sum of money at the outset, and interest on the amount advanced is compounded at a fixed rate. The principal and interest are repaid from the property sale proceeds when the last survivor dies, sells the house, or moves into a long-term care facility permanently. The loan may also be prepaid without a house sale.

Given that the value of the property when the loan is repaid is uncertain, a shortfall in the proceeds from the sale of the home relative to the outstanding mortgage is possible. However, most roll-up mortgages in the UK are sold with the NNEG, which protects the borrower by capping the redemption amount of the mortgage at the lesser of the face amount of the loan and the sale proceeds of the home. Because the interest rate is fixed, borrowers have a financial incentive to repay the loan and refinance when interest rates decline.

In the US, HECMs are the most popular reverse mortgage product, accounting for about 90% of the market share. HECMs are sold to US homeowners who are no younger than 62 years old. In constrast to the UK roll-up mortgages in the UK, most HECMs are originated with an adjustable interest rate linked to the rate on one year Treasury Bills. Also, borrowers tend to choose payments in the form of a line of credit rather than a single lump sum at the outset of the contract.

All HECMs are insured by the Federal Housing Administration of the US Department of Housing and Urban Development. The purposes of this insurance are twofold. The first is to ensure that borrowers will receive cash advances in a timely manner even if their lender becomes bankrupt. The second is to protect lenders from losses if the price of the mortgaged home falls below the loan balance. In this connection, such insurance may be viewed as an embedded guarantee that is similar to the NNEG in the UK.

Each HECM borrower is required to pay a mortgage insurance premium. The current mortgage insurance premium is a front-end charge of 2% of the maximum claim amount,[1] plus a monthly charge of 1/12 of 0.5% of the outstanding loan balance.

2.2 The Embedded Guarantee

We let L_t and H_t be the time-t values of the loan and the mortgaged home, respectively. Suppose that the loan is due at time t. If $H_t \geq L_t$, then the lender will obtain the entire value of the loan, L_t, and the balance of the property price passes to the borrowers or their estate. If $H_t < L_t$, then the lender will obtain only H_t through the NNEG. Mathematically, the repayment to the lender is given by

$$\min(L_t, H_t) = L_t - \max(L_t - H_t, 0), \tag{1}$$

which is the loan value less the payoff from the guarantee. Note that $\max(L_t - H_t, 0)$ is precisely the payoff function for a European put option with a strike price L_t.

The embedded guarantee prevents the borrower from owing more than the value of the mortgaged home when the loan is repaid. The risk that the loan exceeds the home value is borne by the lender (for roll-up mortgages in the UK) or the Federal

[1]The maximum claim amount is the lesser value of the appraised home equity and the maximum loan limit for the geographical area in which the mortgaged property is located.

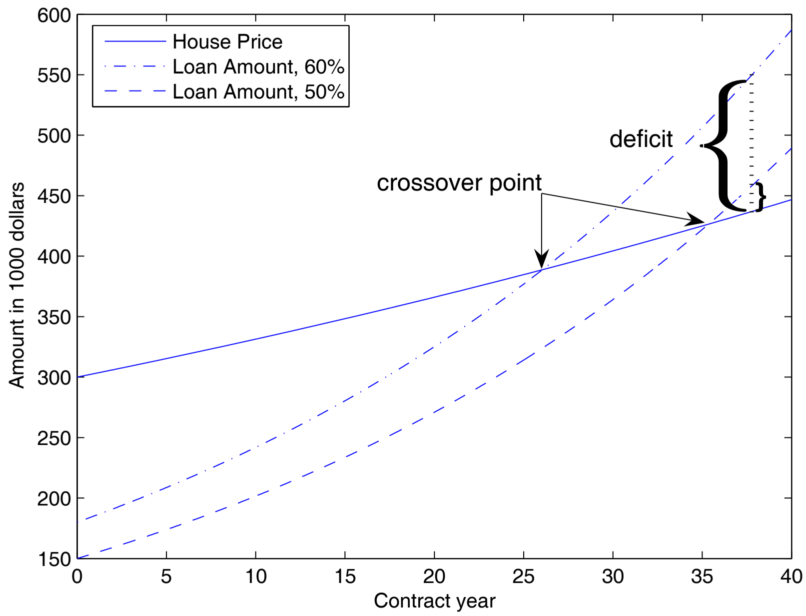

Figure 1: Demonstration of the crossover risk.

Housing Administration (for HECMs in the US). This risk is sometimes referred to the crossover risk.

Let us use a simple roll-up mortgage to illustrate the crossover risk. Assume that the initial value of the mortgaged home is \$300,000 and that the loan to value ratio is 50%. Given the hypothetical trajectory shown in Figure 1 (dashed line), the crossover occurs in about 35 years from now. If the loan is repaid after the crossover, the lender is subject to a loss. If a higher loan-to-value ratio, say 60%, is assumed, the crossover will occur sooner.

From Figure 1 we observe that the crossover risk is affected by the loan-to-value ratio and the interest rate at which the loan is accumulated. The risk is also affected by stochastic factors including the appreciation of house prices and the timing of repayment. Some research on the appropriate model for house prices exists (such as Li et al. (2010)). The focus of this research, though, is the timing of the repayment date, basing the modeling on the semi-Markov model we describe in the following section.

3 A Semi-Markov Multiple State Model

3.1 Modes of Termination

A reverse mortgage may terminate for various reasons.

- Death
  Death is a major mode of termination. Its role is particularly important when the homeowner reaches a very high age.

- Entrance to a long-term care (LTC) facility
  Similar to mortality, health-related moveouts plays a predominant role when the homeowner becomes old.

- Moveout for non-health reasons
  A homeowner may move out his/her mortgaged home permanently for a non-health reason, for example, downsizing.

- Refinancing
  A reverse mortgage may be repaid because of a change in the borrower's financial circumstances. In the UK, voluntary prepayments may be associated with refinancing when the market interest rate is lower than the fixed interest rate at which the loan is accumulated. In the US, refinancing may occur if the younger spouse dies, as the death of the younger spouse as the maximum loan to property ratio is determined as a function of the age of the younger surviving spouse.

3.2 Model Specification

We propose a semi-Markov multiple state model for the terminations. By semi-Markov we mean the transition probability depends not only on the current time and state, but also on the time since the previous transition.

Our model is built upon the work of Ji et al. (2010), who proposed a semi-Markov multiple state model to capture the effect of bereavement.[2] In that paper, the force

[2]The effect of bereavement on mortality is documented in Cox and Ford (1964), Jagger and Sutton (1991) and Young et al. (1963).

of mortality after bereavement is specified by the following parametric functions:

- widows, age x, t years since bereavement:

$$\tilde{\mu}^f(x,t) = (1 + a^f e^{-k^f t})(\mu^f_{x+t} + \lambda) = F^f(t)(\mu^f_{x+t} + \lambda);$$

- widowers, age y, t years since bereavement:

$$\tilde{\mu}^m(y,t) = (1 + a^m e^{-k^m t})(\mu^m_{y+t} + \lambda) = F^m(t)(\mu^m_{y+t} + \lambda).$$

Where λ represents the force of mortality from "common shock" events (events that would cause simultaneous mortality of both lives), and μ^f_{x+t} and μ^m_{y+t} represent the force of mortality for married women and men, respectively, from all causes other than common shock; a^m, a^f, k^m, and k^f are the semi-Markov parameters.

The model assumes that the force of mortality for an individual after bereavement is proportional to the corresponding force of mortality if his/her spouse is still alive. Initially, bereavement increases the force of mortality by $100a^f\%$ for females and $100a^m\%$ for males. The multiplicative factors $F^f(t)$ and $F^m(t)$ decrease exponentially as t increases and finally approach 1. Parameters k^f and k^m govern the speed at which the selection effect diminishes. Further details about this model are given in Ji et al. (2010).

In this work, the model is extended to incorporate additional modes of decrement. The complete specification of our proposed model is shown diagrammatically in Figure 2. The boxes represent the state process during the lifetime of a reverse mortgage. For example, if the process is still in State 0 at time t, that means that both husband and wife are alive at t. In States 1 to 4, only one of joint borrowers is still living at the mortgaged home. In States 5 to 8, the last survivor has either died or permanently left the mortgaged home, and the reverse mortgage is terminated on the transition to any of those states.

The arrows between the states represent the possible transitions, indicating how a reverse mortgage may be terminated. Our model permits transitions from State 0 to 5. This feature captures the simultaneous dependence between joint lifetimes due to common shocks. However, transitions from State 0 to 7 are not permitted, although we can easily relax this assumption if information about long-term care incidence for

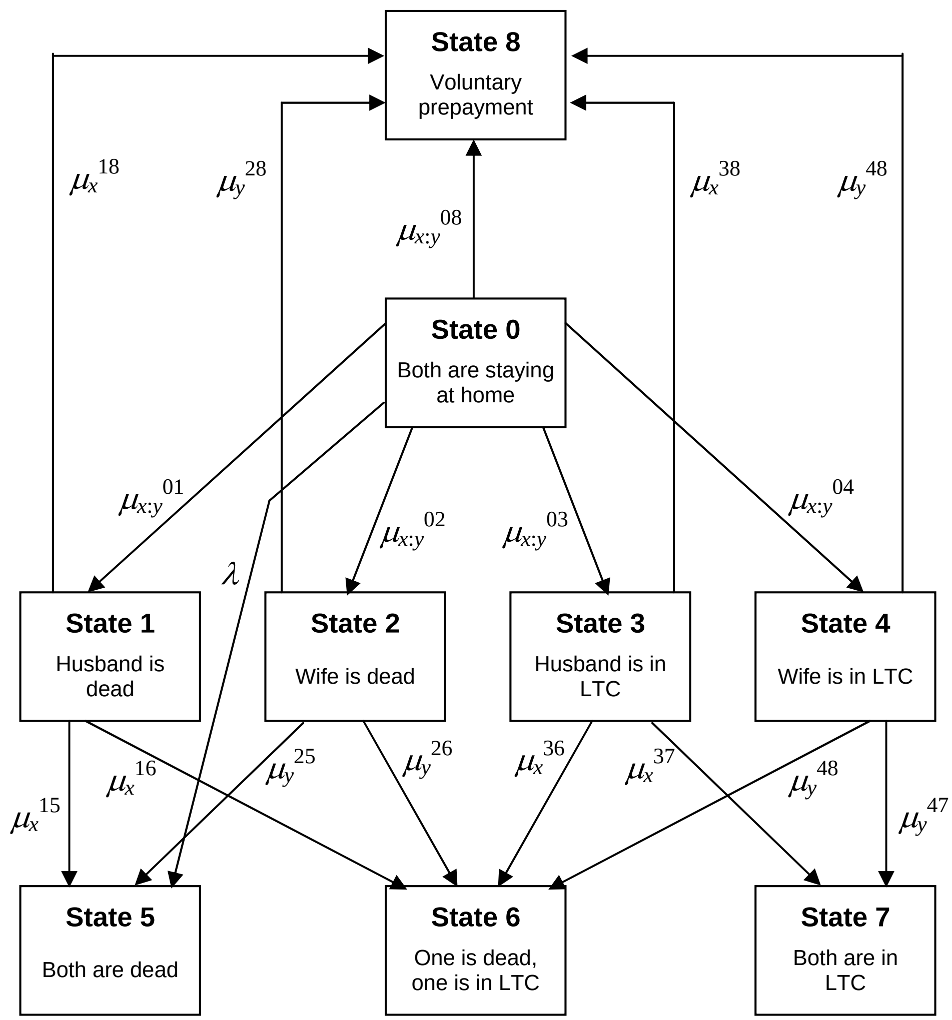

Figure 2: A multiple state model for joint-life reverse mortgages.

married couples is available. For convenience we allocate the two non-health related terminations, moveout for non-health reasons and refinancing, to one single state, State 8, labeled as voluntary prepayment.

Following Ji et al. (2010), a semi-Markovian approach is used to model the effect of bereavement on mortality. Specifically, μ_x^{15} and μ_y^{25} are obtained by multiplying $F^f(t)$ and $F^m(t)$, respectively, with the corresponding pre-bereavement force of mortality.

Let x and y denote the age of a wife and husband, respectively. The forces of transition from State 0, in which both borrowers are living in the mortgaged home, are denoted by $\mu_{x:y}^{0i}$, for $i = 0, 1, 2, 3, 4, 5, 8$. A slightly different notation is used when there is only one person living in the mortgaged home, as we assume the age of each partner is only relevant while that person is still in the home. For example, we use μ_x^{15} to denote the transition intensity for a widow of age x from State 1 to 5. The age of her husband is not included in the notation as it is irrelevant to the calculations.

Since all transitions are unidirectional, it is straightforward to calculate the transition probabilities by using the Kolmogorov forward equations. We denote the probability of transition from State 0 to State i at time t by ${}_tp_{x:y}^{0i}$.

We let ${}_{t|}q_{x:y}^{(\tau)}$ be the probability that a reverse mortgage, issued to a wife aged x and a husband aged y at time 0, is in force at time t and will be terminated before time $t+1$. This aggregate one-year termination probability is of our particular interest, because the simulation studies in later sections are conducted in annual time steps. We can calculate ${}_{t|}q_{x:y}^{(\tau)}$ by summing the probabilities of transition from State 0 to States 5, 6, 7, and 8; that is,

$$
\begin{aligned}
&{}_{t|}q_{x:y}^{(\tau)} \\
= &\int_0^1 {}_tp_{x:y}^{00}\ {}_sp_{x+t:y+t}^{00}\ (\mu_{x+t+s:y+t+s}^{05} + \mu_{x+t+s:y+t+s}^{08})\, ds \\
&+ \int_0^1 {}_tp_{x:y}^{01}\ {}_sp_{x+t:y+t}^{01}\ (\mu_{x+t+s}^{15} + \mu_{x+t+s}^{16} + \mu_{x+t+s}^{18})\, ds \\
&+ \int_0^1 {}_tp_{x:y}^{02}\ {}_sp_{x+t:y+t}^{02}\ (\mu_{y+t+s}^{25} + \mu_{y+t+s}^{26} + \mu_{y+t+s}^{28})\, ds \\
&+ \int_0^1 {}_tp_{x:y}^{03}\ {}_sp_{x+t:y+t}^{03}\ (\mu_{x+t+s}^{36} + \mu_{x+t+s}^{37} + \mu_{x+t+s}^{38})\, ds \\
&+ \int_0^1 {}_tp_{x:y}^{04}\ {}_sp_{x+t:y+t}^{04}\ (\mu_{y+t+s}^{46} + \mu_{y+t+s}^{47} + \mu_{y+t+s}^{48})\, ds.
\end{aligned}
$$

4 Valuing NNEGs in the UK

4.1 The Pricing Formula

Let us define the following notation:

- r: the continuously compounded risk-free interest rate;

- g: the continuously compounded rental yield;
- u: the continuously compounded roll-up interest rate on the loan;
- L_t: the value of the reverse mortgage loan at time t, excluding NNEG; $L_t = L_0 e^{ut}$;
- H_t: the value of the mortgaged property at time t;
- δ: the average delay in time from the point of home exit until the actual sale of the property.
- c: the cost (as a percentage of the property value) associated with the sale of the property;
- ω: the highest attained age;
- $P(t, S_0, K, r, g, \sigma)$: the time-zero value of a put option on an asset with initial value S_0, volatility σ and dividend yield g; the option matures at time t and has a strike price K.

We use discrete annual time steps and assume that all decrements occur at mid-year, the value of a NNEG written to a wife of age x and a husband of age y can be expressed as

$$\sum_{t=0}^{\omega-\min(x,y)-1} P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0 e^{ut}, r, g, \sigma\right) \ {}_{t|}q_{x:y}^{(\tau)}, \tag{2}$$

where ${}_{t|}q_{x:y}^{(\tau)}$ is the probability that the loan is terminated between year t to year $t+1$. This probability is calculated on the basis of the semi-Markov multiple state model.

In our illustrations we assume that property prices follow a geometric Brownian motion. The same assumption on house prices is also used by the Institute of Actuaries (2005b). A more realistic econometric model may be used, but we do not explore this aspect of the problem. See Li al al. (2010) for more discussion of house price modeling.

Assuming that the price of the mortgaged property follows a geometric Brownian motion, $P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0 e^{ut}, r, g, \sigma\right)$ can be calculated by the Black-Scholes formula:

$$L_0 e^{(u-r)(t+\frac{1}{2}+\delta)} \ N(-d_2) \ - \ H_0(1-c) e^{-g(t+\frac{1}{2}+\delta)} \ N(-d_1), \tag{3}$$

Parameter	Assumed value
r	4.75%
g	2%
u	7.5%
δ	0.5 year
c	2.5%
L_0	£30 000

Table 1: Assumed values of the parameters in the NNEG pricing formula.

Age of the younger spouse at inception	60	70	80	90
Initial house value	£176 500	£111 000	£81 000	£60 000

Table 2: Minimum initial house values. Source: Institute of Actuaries (2005b).

where $d_1 = \frac{\ln\left(\frac{H_0(1-c)}{L_0}\right)+(r-u-g+\frac{\sigma^2}{2})\ (t+\frac{1}{2}+\delta)}{\sigma\sqrt{t+\frac{1}{2}+\delta}}$, and $d_2 = d_1 - \sigma\sqrt{t+\frac{1}{2}+\delta}$.

In practice, the roll-up rate u is greater than the risk-free interest rate r. When $u > r$, the value of $P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0e^{ut}, r, g, \sigma\right)$ will be a strictly increasing function of t. Therefore, decrement assumptions play a critical role in valuing the guarantee.

The assumed parameter values are summarized in Table 1. The initial house value H_0 is the minimum assumed starting property value required for a loan of $L_0 = £30\,000$ (see Table 2). Note that the value of H_0 is positively related to the age of the younger spouse at inception – older lives may borrow more, because of the reduced crossover risk.

4.2 The Impact of Mortality

Here we examine the impact of mortality assumptions on the value of a NNEG. For now we assume that death is the only mode of decrement.

Our analysis is based on the joint-life mortality data considered by Frees et al. (1996), Youn and Shemyakin (2002), and Ji et al. (2010). The data comprise 14,947 records of joint and last-survivor annuity contracts written by a large insurance company in Canada. It seems reasonable to assume that annuitants and people who participate in reverse mortgages will have similar mortality profiles.

First we price guarantees written to single lives. Using the joint-life mortality data, we estimate the marginal survival distribution for each gender. Termination probabilities for all durations are calculated accordingly and are substituted into equation (2) to obtain the guarantee value. The relationship between the value of the guarantee and the age at inception is depicted in Figure 3. The guarantee values for females are higher than the corresponding values for males, because female mortality is generally lighter than male mortality.

Next we price guarantees written to joint lives. To examine how the value of a NNEG may be affected by the dependence between two lifetimes, we use two different assumptions. First, we assume independence between the lifetimes of the husband and wife, and use the marginal distributions. Secondly, we use the semi-Markov multiple state model (with States 0, 1, 2, and 5), which is fitted to the same data set. We use simulation (with 100,000 projections) to estimate aggregate termination probabilities, which are then applied to equation (2) to obtain the guarantee value.

In Figure 3 we show the guarantee values calculated with both assumptions. For joint lives, the x-axis in the diagram corresponds to the age of the wife, who is assumed to be two years younger than the husband. From the diagram we observe that the assumption of independence generally leads to an overestimation of NNEG prices. The overestimation is especially significant at high ages. This may be explained by the semi-Markov property, which allows widows and widowers to recover from bereavement. In particular, as younger widowed borrowers have time to recover, the impact of bereavement on the guarantee value is relatively low. The opposite is true for older widowed borrowers.

4.3 The Impact of Long-Term Care Incidence

We now study the impact of long-term care incidence on the value of a NNEG. The model used here is comprised of States 0 to 7, assuming that mortality and entrance to a long-term care facility are the only two modes of termination.

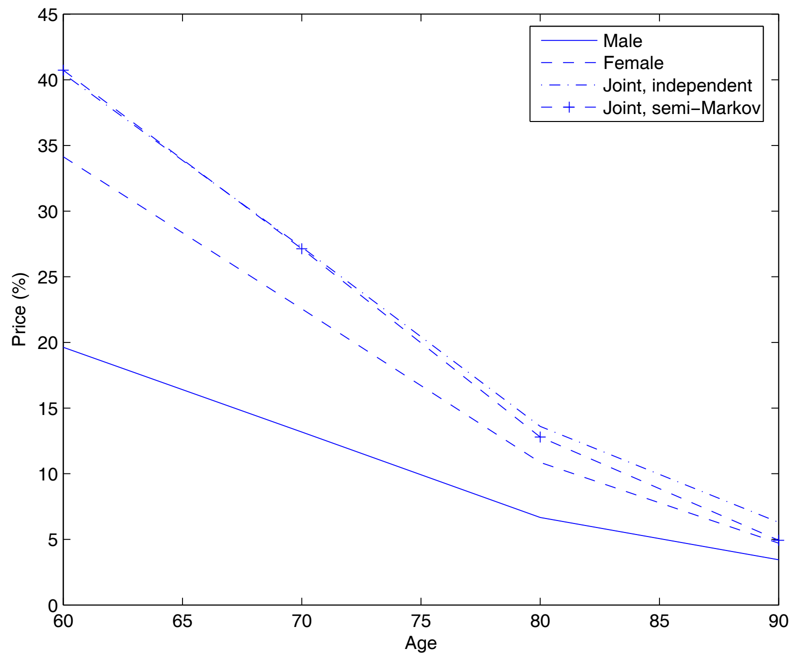

Figure 3: Simulated NNEG prices (as a percentage of cash advanced), single lives and joint lives with a 2-year age gap.

Generally speaking, people living in long-term care facilities are less healthy than those remaining in their own homes. This means that the introduction of long-term care incidence to the model impacts the at-home mortality. Therefore, besides estimating the additional forces of transition, the forces of transition that are included in the model considered in Section 4.2 must also be re-estimated or at least adjusted.

We derive the required forces of transition by a proportional adjustment. Let μ_x^f and μ_y^m be the forces of mortality (from all causes other than common shock) for a wife of age x and a husband of age y, respectively (see Section 3.2). We model the forces of transition to a long-term care facility for males and females by $\rho^m \mu_y^m$ and $\rho^f \mu_x^f$, respectively, where ρ^m and ρ^f are constants. We model the knock-on impact by assuming males and females at-home mortality are obtained by multiplying their forces of mortality (from all causes other than common shock) with constant proportional factors θ^m and θ^f, respectively.

We assume that bereavement has an effect on the forces of transition from States 1 and 2. The forces of transition from these states are obtained on the basis of the

When both borrowers are alive
$\mu_{x:y}^{01} = \theta^m \mu_y^m$
$\mu_{x:y}^{03} = \rho^m \mu_y^m$
$\mu_{x:y}^{02} = \theta^f \mu_x^f$
$\mu_{x:y}^{04} = \rho^f \mu_x^f$
$\mu_{x:y}^{05} = \lambda$
When one of the borrowers is dead (s years after bereavement)
$\mu_x^{15} = F^f(s)(\theta^f \mu_x^f + \lambda)$
$\mu_x^{16} = F^f(s)(\rho^f \mu_x^f)$
$\mu_y^{25} = F^m(s)(\theta^m \mu_y^m + \lambda)$
$\mu_y^{26} = F^m(s)(\rho^m \mu_y^m)$
When one of the borrowers is in a long-term care facility
$\mu_x^{36} = \theta^f \mu_x^f + \lambda$
$\mu_x^{37} = \rho^f \mu_x^f$
$\mu_y^{46} = \theta^m \mu_y^m + \lambda$
$\mu_y^{47} = \rho^m \mu_y^m$

Table 3: Transition intensities for the model in Section 4.3.

semi-Markov functions $F^f(t)$ and $F^m(t)$ defined in Section 3.2. For instance, s years after bereavement, the force of transition from State 1 to 5 is given by $F^f(s)(\theta^f \mu_x^f + \lambda)$. The expressions for all forces of transition in the model are provided in Table 3.

To estimate the proportional factors for the move from an aggregate model to the at-home/in LTC split model, we make use of the findings in the Equity Release Report of the Institute of Actuaries (2005b). In the report, the following ratios are provided:

- L_y^m: long-term care incidence rate to population mortality rate (males, age y);
- L_x^f: long-term care incidence rate to population mortality rate (females, age x);
- A_y^m: at-home mortality to population mortality rate (males, age y);
- A_x^f: at-home mortality to population mortality rate (females, age x).

Given these ratios, the proportional factors are calculated by solving the following

Age	ρ^m	θ^m	ρ^f	θ^f
≤ 70	0.05	0.97	0.10	0.95
80	0.07	0.97	0.20	0.90
90	0.15	0.94	0.33	0.85
≥ 100	0.22	0.94	0.46	0.80

Table 4: Estimate values of ρ^m, θ^m, ρ^f, and θ^f.

equations numerically:

$$\begin{aligned}
L_y^m &= \frac{\int_0^1 e^{-\int_0^s (\theta^m+\rho^m)\mu_{y+u}^m+(\theta^f+\rho^f)\mu_{y+u}^f+\lambda du}\ \rho^m\mu_{y+s}^m\, ds}{\int_0^1 e^{-\int_0^s \mu_{y+u}^m+\mu_{y+u}^f+\lambda du}\ (\mu_{y+s}^m+\lambda)\, ds};\\
A_y^m &= \frac{\int_0^1 e^{-\int_0^s (\theta^m+\rho^m)\mu_{y+u}^m+(\theta^f+\rho^f)\mu_{y+u}^f+\lambda\, du}\ (\theta^m\mu_{y+s}^m+\lambda)\, ds}{\int_0^1 e^{-\int_0^s \mu_{y+u}^m+\mu_{y+u}^f+\lambda du}\ (\mu_{y+s}^m+\lambda)\, ds};\\
L_x^f &= \frac{\int_0^1 e^{-\int_0^s (\theta^m+\rho^m)\mu_{x+u}^m+(\theta^f+\rho^f)\mu_{x+u}^f+\lambda du}\ \rho^f\mu_{x+s}^f\, ds}{\int_0^1 e^{-\int_0^s \mu_{x+u}^m+\mu_{x+u}^f+\lambda du}\ (\mu_{x+s}^f+\lambda)\, ds};\\
A_x^f &= \frac{\int_0^1 e^{-\int_0^s (\theta^m+\rho^m)\mu_{x+u}^m+(\theta^f+\rho^f)\mu_{x+u}^f+\lambda\, du}\ (\theta^f\mu_{x+s}^f+\lambda)\, ds}{\int_0^1 e^{-\int_0^s \mu_{x+u}^m+\mu_{x+u}^f+\lambda du}\ (\mu_{x+s}^f+\lambda)\, ds}.
\end{aligned}$$

Note that in equations above, it is assumed that the husband and wife are of the same age. In Table 4 we show the estimated proportional factors for ages ≤ 70, 80, 90, and ≥ 100. For ages 71-79, 81-89 and 91-99, the proportional factors are obtained by linear interpolation.

We then use the eight-state model to simulate prices of NNEGs written to joint-borrowers. To examine the impact of long-term care incidence on the NNEG prices, we consider the following four cases:

- Case 1: Central estimate of ρ.
- Case 2: ρ is increased by 30%, other things equal.
- Case 3: ρ is decreased by 30%, other things equal.
- Case 4: No long-term care incidence.

Age of wife at inception	60	70	80	90
Case 1	36.19%	23.57 %	10.51%	3.85%
Case 2	34.22%	22.09 %	9.57%	3.43%
Case 3	38.47%	25.36 %	11.64%	4.39%
Case 4	40.73%	27.13 %	12.79%	4.93%

Table 5: Simulated NNEG prices (as a percentage of cash advanced) under different assumptions about long-term care incidence.

Table 5 shows the NNEG prices under the four assumptions about long-term care incidence. It is assumed in the calculations that the husband is two years older than the wife. From Table 5 we observe that, when long-term care incidence is introduced to the model, the resulting guarantee value will drop by 11% and 22% for young-old borrowers and old-old borrowers, respectively. This indicates that entrance to a long-term care facility is a significant mode of termination, and that it requires adequate modeling.

In expressing long-term care incidence rates as a fraction of the corresponding base mortality rates, we have implicitly assumed that the desire to move to a long-term care facility is determined by age-related health conditions only. However, other factors, for example, the quality of long-term care facilities, may also affect one's desire to move. If data permits, such factors may be incorporated into the model by modifying the expressions for the forces of transition accordingly.

4.4 The Impact of Voluntary Prepayment

We now consider the full nine-state model, which incorporates three modes of termination including voluntary prepayment. A roll-up mortgage may be prepaid voluntarily due to a non-health related moveout or refinancing.

There is little information about non-health related moveouts available in the public domain. In our calculations, we use the assumptions made by the the Institute of Actuaries (2005b), which are summarized in Table 6. The initial selection effect is modeled by using lower rates for early contract years. We assume further that, after the fifth contract year, the rate of non-health related moveouts is reduced by

Contract year	Prepayment rate
1	0.0%
2	0.0%
3	0.15%
4	0.3%
5	0.3%
6+	0.75%

Table 6: Allowances for prepayments arising from changes in personal circumstances expressed as a percentage of in force contracts. Source: Institute of Actuaries (2005b).

Contract year	Remortgaging rate
1-2	1.0%
3	2.0%
4-5	2.5%
6-8	2.0%
9-10	1.0%
11-20	0.5%
21+	0.25%

Table 7: Assumed remortgaging rates. Source: Hosty et al. (2007).

0.25% if both lives are still staying in the mortgaged property. This is assumes that a borrower may have a greater desire to move out after his/her spouse has died.

As the roll-up interest rate is usually fixed, borrowers may have a financial incentive to refinance when there is a fall in market interest rates. Here we use the remortgaging rates assumed by Hosty et al. (2007), which they claim to be suitable for reverse mortgages sold at a time when interest rates are relatively low but not bottom of the market.[3] The assumed remortgaging rates are displayed in Table 7.

We incorporate these voluntary prepayment rates into the full nine-state model

[3]More specifically, Hosty et al. (2007) claim that the remortgaging rates in Table 7 might be considered best estimates for a provider with robust early repayment charges distributing a flexible product at competitive but not market-leading rates through a broker distribution channel at a time when interest rates are relatively low but not bottom of the market (say headline rates of 6.5% p.a.).

Age of wife at inception	60	70	80	90
Case 1	24.43%	17.09%	8.15%	3.28%
Case 2	21.77%	15.52%	7.57%	3.13%
Case 3	27.49%	18.83%	8.80%	3.44%
Case 4	36.19%	23.57%	10.51%	3.85%

Table 8: Simulated NNEG prices (as a percentage of cash advanced), under different assumptions about voluntary prepayments

to price a NNEG. To examine the impact of the assumption, we consider four cases:

- Case 1: Central rates of voluntary prepayment from Table 7.
- Case 2: The rates of voluntary prepayment are increased by 30%, other things equal.
- Case 3: The rates of voluntary prepayment are reduced by 30%, other things equal.
- Case 4: No voluntary prepayment.

The simulated NNEG prices for all four cases are shown in Table 8. It is assumed in the calculations that the husband is two years older than the wife. Note that the prices for Case 4 are taken from Section 4.3. From Table 8 we observe that the NNEG prices drop significantly when voluntary prepayment is taken into account. The effect of voluntary prepayment is even more significant than the effect of long-term care incidence.

The analysis shows that, using reasonable assumptions for the termination model, the impact of health and non-health related move-outs is very significant, and that the semi-Markov model offers a transparent and flexible approach to the modelling of terminations.

5 Valuing HECM Insurance Premiums in the US

5.1 The Pricing Formula

In this section, we use the model and parameters developed above to analyze the premium that US HECM borrowers pay for the NNEG. All HECMs sold in the US are insured by the Federal Housing Agency, with premiums paid by the borrowers. For a loan written to a wife of age x and a husband of age y, the time-0 value of the mortgage insurance can be expressed as follows:

$$\sum_{t=0}^{\omega-\min(x,y)-1} P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0 e^{ut}, r, g, \sigma\right)\ {}_{t|}q_{x:y}^{(\tau)} \tag{4}$$

where L_0 is the amount borrowed, including the origination costs and front-end mortgage insurance premium; H_0 is the adjusted property value when the loan is originated. We assume that H_0 is always smaller than the HECM maximum loan limit for the area in which the property is located. Other symbols in the above expression carry the same meaning as they do in Section 4.1. It is assumed in our calculations that all loans terminate at mid-year.

The current mortgage insurance premium is a front-end charge of 2% of the home value plus a monthly premium of 1/12 of 0.5% of the outstanding loan balance. The current front-end charge of 2% has been criticized for being too high, for example in Caplin (2002).

Let α be the fair front-end charge, expressed as the a percentage of the home value. The time-0 value of the total mortgage insurance premium can be expressed as

$$\alpha\ H_0 + 0.005 \times \frac{1}{12} \sum_{k=0}^{\omega-\min(x,y)-1} \left({}_{k|}q_{x:y}^{(\tau)} \sum_{t=1}^{12(k+\frac{1}{2})} L_0 e^{\frac{(u-r)t}{12}} \right). \tag{5}$$

By actuarial equivalence, we have the following formula for calculating α:

$$\begin{aligned} \alpha \quad = \quad & \frac{1}{H_0} \sum_{k=0}^{\omega-\min(x,y)-1} \left\{ {}_{k|}q_{x:y}^{(\tau)} \left[P\left(k+\frac{1}{2}+\delta, H_0(1-c), L_0 e^{ut}, r, g, \sigma\right) \right.\right. \\ & \left.\left. -0.005 \times \frac{1}{12} \sum_{t=1}^{12(k+\frac{1}{2})} L_0 e^{\frac{(u-r)t}{12}} \right] \right\}. \end{aligned} \tag{6}$$

Age of the younger spouse at inception	65	70	75	80	85	90
Principal limit factor	0.565	0.605	0.648	0.691	0.733	0.772

Table 9: Principal limit factors for HECM loans in 90803 Los Angeles.

Here we assume again that house prices follow a geometric Brownian motion.

When H_0 is smaller than the HECM maximum loan limit, the maximum amount that a borrower can borrow is the product of H_0 and the applicable principal limit factor, f, which depends on the expected interest rate and the borrower's age at inception. For example, if the applicable principal limit factor is 0.551 and the value of the mortgaged property is \$100,000 at inception, then the maximum amount that can be borrowed would be \$55,100, including the origination costs paid on the borrower's behalf.

Table 9 displays the principal limit factors for different ages of the younger spouse at inception. These factors, which are applicable to HECM loans with monthly adjusted interest rate, are obtained from the online reverse mortgage calculator provided on Wells Fargo Bank's website[4] on 13 May 2010. The calculations were based on zip code 90803 Los Angeles, which has the highest number of HECM loans. Our calculations will be based on these principal limit factors.

However, most borrowers do not use the full principal limit. According to the US Department of Housing and Urban Development (2008), over three quarters of the borrowers chose the line of credit payment option and 12% of the borrowers chose the payment option with monthly payments and a reduced line of credit. Although most borrowers use a sizeable amount of their lines of credit at the inception of the contract, they usually do not exhaust the line of credit during the term of the loan.

If the borrower withdraws $100\phi\%$ of the maximum amount that he/she can borrow, then L_0 in equation (6) can be expressed as $\phi f H_0$. Furthermore, we can show easily that α does not depend on H_0 if we assume house prices follow a geometric Brownian motion.

In our calculations, the following three scenarios are considered: $\phi = 1$, $\phi = 0.9$, and $\phi = 0.8$. The assumed values for other model parameters are described below.

[4]https://www.benefits-mortgage.com/calculator/entry.do?linkType=mps.

- The house price volatility, σ, is 12%. This is the historical volatility of the Quarterly Purchase-only House Price Index from 1991 to 2008 provided by Office of Federal Housing Enterprise Oversight.

- The continuously compounded risk-free interest rate, r, is 3.49%. This is 10-year US Treasury rate on 13 May 2010, obtained from the website of the US Department of Treasury.[5]

- The continuously compounded roll-up rate, u, is 2.37%. This is sum of the one year constant maturity Treasury rate of 0.40% on 13 May 2010, a lender spread margin of 1.5%, and a mortgage insurance premium of 0.5% (see, e.g., Rodda (2004, p.593)).

As in pricing the UK NNEGs, we assume that $g = 2\%$, $c = 2.5\%$, and $\delta = 0.5$.

5.2 Decrement Assumptions

When we apply the semi-Markov multiple state model to HECM insurance premiums, the following assumptions are used:

- Mortality and long-term care incidence

  Central assumptions about mortality and long-term care incidence are the same as those used in Section 4.

- Refinancing

  In contrast to roll-up mortgages sold in the UK, most HECMs are floating rate mortgages, which means borrowers have rather low incentive to refinance. The current market practice uses the age of the younger spouse to determine the principal limit of the loan. Therefore, a borrower may remortgage when his/her spouse dies, as that may lead to an increase in the principal limit. However, such a refinancing arrangement would not affect the existing mortgage insurance. In this connection, refinancing is not considered when we compute HECM insurance premiums.

[5] http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml.

- Non-health-related moveouts

  In a US-specific study on mobility, Zhai (2000) argued that mobility is a combined result of increasing health-related and declining non-health-related moveouts, plus a static rate. Zhai went on to derive an U-shaped curve of mobility rates, which says that the rate of mobility (for both health and non-health related reasons) declines from 4.8% at age 60 to 3.2% at age 80, and then rises slowly to about 4.2% at age 105.

  We set our assumptions about non-health related moveouts on the basis of the U-shaped curve provided by Zhai (2000). In particular, since mobility at younger ages is mostly non-health related, we assume that the rate of non-health related moveout is 4% from age 60 to 65. This rate is linearly reduced to 1% for age 90 and above, when mobility is mostly health-related. Following Zhai (2000), we discount the mobility rate by 50% when both borrowers are living in the mortgaged property.

  We further model the effect of selection by applying a 80% discount to the mobility rate during the first contract year. The discount is reduced linearly to zero during the tenth contract year.

Having established the decrement assumptions, we can simulate the survival curve for a HECM contract. From the survival curve we can tell the probability that the HECM contract is still in force at a certain time after inception.

In Figure 4 we show the survival curves for a HECM contract written to a 62-year-old wife and a 64-year-old husband, when different modes of termination are incorporated into the model. We observe from the diagram that long-term care incidence and voluntary prepayments (non-health related moveouts) would significantly reduce the survival probabilities, thereby shortening the expected duration of the HECM contract.

5.3 The Estimated Premiums

In Table 10 we display the estimated values of α (the front-end charge as a percentage of house value) in various scenarios. The calculations are based on 100,000 simulations from the multiple state model and the assumption that the husband is two years

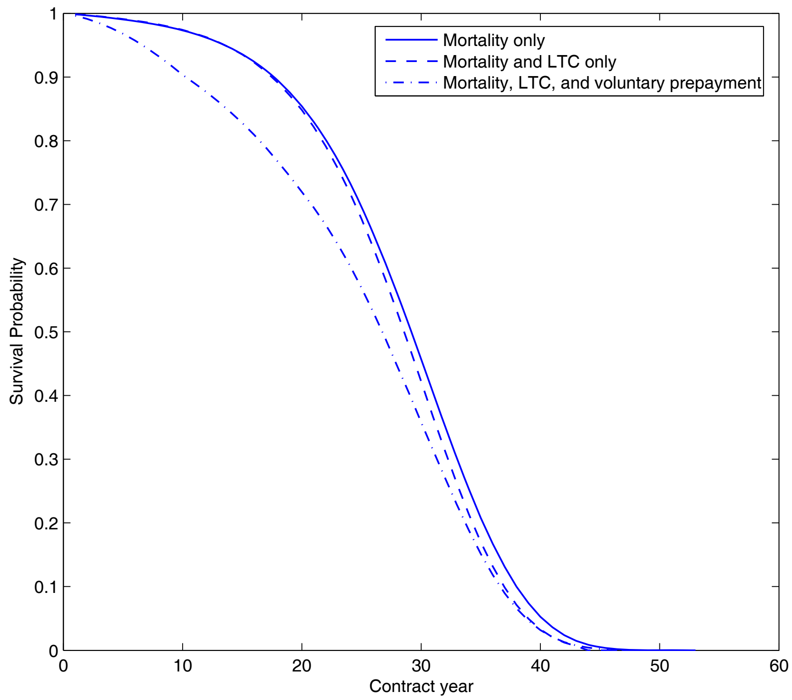

Figure 4: Estimated survival curves for a HECM contract written to a 62-year-old wife and a 64-year old husband.

older than the wife. We observe that the estimated fair front-end charge will drop significantly if the joint borrowers do not exhaust the available line of credit when the loan is originated. When 80% of the principal limit is advanced, the required front-end charges are negative in all cases we consider. This means that the monthly premium itself is more than enough to cover the cost of the embedded guarantee.

Given the model and the assumptions describe, our work supports Caplin(2002) in finding that the current front-end charge appears too high, even if the borrower withdraws the maximum loan. This suggests that the front-end charge could be lowered to make HECMs more attractive to potential borrowers. However, we note that the GBM assumption for house prices will (probably) lead to an underestimate of the guarantee cost, given that the process is more likely to exhibit fatter tails and autocorrelation.

The simulation results point to two other important conclusions. First, since

Age of wife at inception	65	70	75	80	85	90
Scenario 1: $\phi = 1$						
Front-end premium	-0.74%	-0.12%	0.52%	1.06%	1.45 %	1.63%
Scenario 2: $\phi = 0.9$						
Front-end premium	-1.56%	-1.13%	-0.68%	-0.29%	0.01%	0.16%
Scenario 3: $\phi = 0.8$						
Front-end premium	-2.13%	-1.81%	-1.48%	-1.16%	-0.88%	-0.67%

Table 10: Simulated front-end premiums for different values of ϕ.

the estimated front-end charge increases with the borrowers' ages, younger couples are subsidizing older couples under the current premium structure. The problem could be ameliorated by using an age-dependent front-end charge, or an adjusted age-dependent principal limit factor may be offered to younger borrowers, as this, according to our results, does not seem to undermine the long-term financial soundness of the HECM insurance fund. Higher principal limit factors may encourage households to participate in reverse mortgages. We expect that this change will be profound, since the US reverse mortgage market has seen a shift to younger elderly homeowners (Bishop and Shan, 2008).

Secondly, the estimated front-end charge decreases dramatically when the utilization of the available line of credit is reduced. This implies that, under the current premium structure, borrowers who utilize a smaller portion of the available line of credit are subsidizing those who utilize more. This problem can be understood more easily from Figure 5, in which we plot, for three different degrees of utilization, the ratio of HECM's current front-end charge to total time-0 value of the embedded guarantee. We observe from this diagram that the problem of overcharging is particular severe if a smaller fraction of the principal limit is withdrawn. A fairer premium structure would use a multiple of the loan utilised rather than the house value as the basis for the front-end charge.

Finally, we calculate the front-end charge for three typical age combinations: (1) the husband and wife are of the same age; (2) the husband is 1 year older than his wife; (3) the husband is 3 years older than his wife. The calculations are based on 100,000 simulations from the termination model and the assumption that $\phi = 1$. The results, which are displayed in Table 11, indicate that the front-end charge is quite dependent

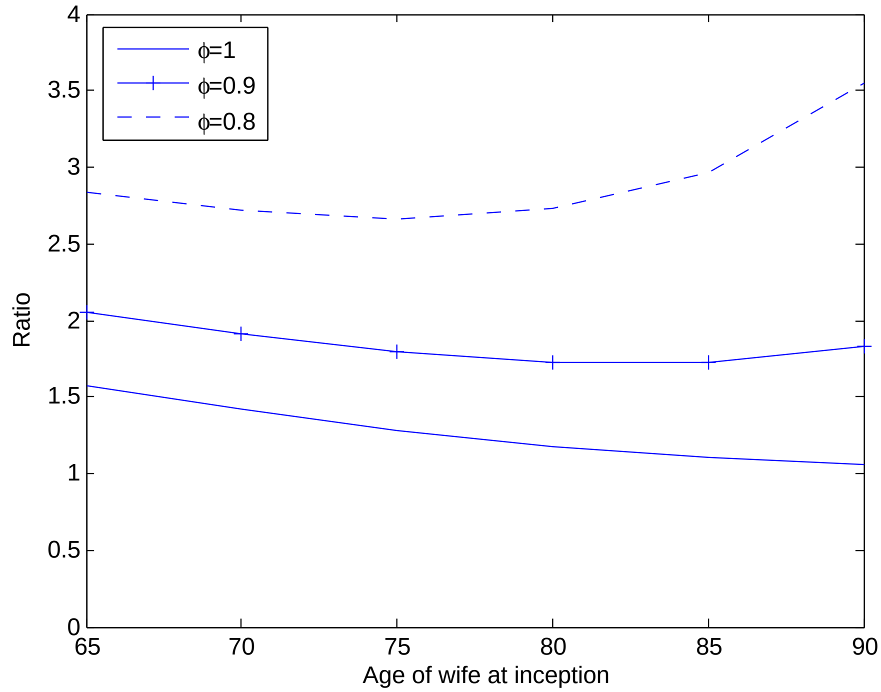

Figure 5: Ratios of the current front-end charge to the total guarantee value when different portions of the available line of credit are utilised.

on the age gap. This effect arises from the dependence among joint-lives, suggesting the adavantage of using a joint-life termination model when valuing joint-life reverse mortgages.

6 Concluding Remarks

In this research, we have proposed a semi-Markov multiple state model for reverse mortgage terminations. The model incorporates not only multiple modes of decrement, but also the statistical dependence between the lifetimes of a husband and wife. This feature is particularly important for valuing joint-life reverse mortgages, which have become increasingly popular in recent years.

Because most data on reverse mortgage terminations are proprietary, some proxy data and assumptions are used in our illustrations. Nevertheless, this does not affect our objectives, to demonstrate how the model can be used to determine a fair price for the NNEG, and to demonstrate the relative importance of different modes of ter-

Age of wife at inception	65	70	75	80	85	90
Scenario 1: the husband and wife are of the same age						
front-end premium	-0.72%	0.01%	0.56%	1.14%	1.55%	1.76%
Scenario 2: the husband is 1 years older than the wife						
front-end premium	-0.73%	-0.11%	0.54%	1.10%	1.50%	1.70%
Scenario 3: the husband is 3 years older than the wife						
front-end premium	-0.74%	-0.13%	0.50%	1.03%	1.41%	1.57%

Table 11: Simulated front-end premiums for different age combinations.

mination. Reverse mortgage providers, who have access to their own decrement data, can easily adapt the multiple state model we propose to suit their own experience.

When the model was applied to HECMs in the US, it was found that, in today's interest rate environment, the current front-end mortgage insurance premium is excessively high. This is consistent with the fact that HECM has always maintained a negative credit subsidy rate.[6] Our findings indicate that there may be room to reduce the front-end charge, particularly to younger borrowers.

What determines the claim from a HECM mortgage insurance is the value of the mortgaged property when the loan is due, usually many years from the time when the loan is written. Hence, in some sense, the heavily front-loaded mortgage insurance premium means that HECMs have front-loaded revenue and back-loaded risk. From a risk management viewpoint, an alternative premium structure with a lower front-end charge would seem to be more effective for capturing the risk associated with the uncertainty in future house prices.

Readers should keep in mind that our conclusions on HECM mortgage insurance premiums are based on the interest rate and the principal limit factors as of this writing. When interest rates change, the principal limit factors, and hence the estimated mortgage insurance premiums, will change accordingly. It is important to take the change in interest rates, possibly through a stochastic interest rate model, into

[6]Credit subsidy represents the projected net present values of all cashflows (premium inflows as well as insurance claim outflows) associated with new loan guaranty commitments over the life of these loans. We refer readers to the US Department of Housing and Urban Development (2008) for further information about the credit subsidy associated with HECM insurance guarantees.

account when deciding a new premium structure. The use of a model such as ours to determine principal limit factors could improve the product design,

We repeat two caveats around the specific numerical results presented here. The geometric Brownian motion assumption for house prices is probably too thin tailed. In practice, one may consider a house price model that permits autocorrelation and stochastic volatility. For example, Li et al. (2010) fit an ARMA-EGARCH model for house prices in the UK; Chen et al. (2009) use an ARMA-GARCH model for house prices in the US. The use of such models will imply market incompleteness, which adds an extra challenge in the pricing process.

The loan interest rate is another variable that we have not explored extensively. It will, of course, affect how fast a floating rate loan is accumulated. It will also affect the guarantee values through the correlation with house prices, as we have observed painfully through the recent financial crisis. Furthermore, there will be dependence between the termination transition probabilities, especially for the non-health related terminations. For example, it is more likely that a borrower will move and repay his/her reverse mortgage in a booming economy. In a recession, homeowners may be less likely to choose the expensive option of long term care. In future research, it would be interesting to integrate a stochastic interest rate model into the multiple state termination model, possibly through a regime-switching framework.

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