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MATCHING AND PORTFOLIO SELECTION: PART 1

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1. INTRODUCTION

1.1 My paper entitled 'The Matching of Assets to Liabilities'[1] described a new study of the subject of matching. The paper outlined the mathematical framework, showed some calculations, and suggested various applications in the field of actuarial valuation.

1.2 Subsequently A. D. Wilkie published a note entitled 'Portfolio Selection in the Presence of Fixed Liabilities: a Comment on "The Matching of Assets to Liabilities" '.[2] Wilkie's note appears to represent a new extension of portfolio selection theory which enables him to encompass the conventional portfolio theory and to show where my approach to matching fits into the picture.

1.3 This paper is in the nature of a reply to that of Wilkie. My objective at the stage of the first draft was to re-examine the matching portfolio, as defined in relation to specified liabilities, in view of the wide range of alternative portfolios which Wilkie has described. The scope of this work progressed well beyond my original plans when I found a most interesting mathematical relationship between the 'matching portfolios' described in my earlier paper and the 'efficient portfolios' described in Wilkie's.

1.4 The analysis which follows is based on the same stochastic type of actuarial model as was used for those two previous papers. The elements of the model are:

(a) a portfolio of assets with specified future cash flows;
(b) a set of liabilities with specified future cash flows;
(c) a statistical model for uncertain factors such as future rates of investment returns and inflation;
(d) the ultimate surplus from the portfolio after all the liability payments have been met, which can be regarded as a random variable;
(e) the mean of the ultimate surplus; and
(f) the variance of the ultimate surplus about its mean.

1.5 For the purpose of general description in this paper the mean ultimate surplus (*e*) will be called the 'expected return' and the variance of ultimate surplus (*f*) will be called the 'risk'. The expected return and risk of a portfolio are therefore being defined relative to specific liabilities. If the liabilities change, so do the risk and expected return on the given portfolio. I shall also define a related measure of risk which I call the 'degree of risk' (see § 3.11).

1.6 The mathematical relationship derived in this paper shows how any efficient portfolio can be separated into three mutually exclusive and distinct components:

(i) the matching portfolio, which is defined by the property that the expected ultimate surplus, or return, is zero and the variance of ultimate surplus, or risk, is minimized;

(ii) a component which is related to the expected return on the portfolio but not to its degree of risk; and

(iii) a component which is related to the degree of risk in the portfolio but not to its expected return.

2. PORTFOLIO SELECTION IN THE PRESENCE OF FIXED LIABILITIES

2.1 The techniques of portfolio selection and modern portfolio theory (as described, for example, by Moore[3]) do not seem to deal comprehensively with the long term liabilities of an investing institution such as a pension trust or a life office. It is true that principles of asset selection can be applied for such institutions by treating the liabilities as a negative asset of which the portfolio must contain unit amount. However the price of the negative asset, that is the value of the liabilities, must be specified. The specification of the price requires an actuarial valuation of the liabilities and it is in the nature of any actuarial valuation that it requires some view to be taken as to the assets which are held or which are deemed appropriate to be held against the liabilities. We then have a circular line of reasoning because the selection of a portfolio in the presence of liabilities depends upon a prior view regarding the assets.

2.2 Wilkie[2] has broken the circular argument. His model, like mine, assumes random rates of return on the future re-investment of cash flows. He analyses portfolios according to their resulting characteristics measured in terms of three quantities:

P: the aggregate present price of all the assets in the portfolio;

E: the expected ultimate surplus of assets net of liabilities on completion of the liability cash flows, (i.e. the expected return); and

V: the variance of the ultimate surplus (i.e. the risk).

2.3 Wilkie has therefore generalized conventional portfolio theory by including the price P of the portfolio as a third dimension. In the conventional theory only E and V are considered because, in the absence of fixed unmarketable liabilities, the proportions of assets to be held in the selected portfolio will be the same whatever the value of P.

2.4 Wilkie describes the concept of 'efficient portfolios' in the context of his three dimensional P–E–V analysis. He illustrates the range of efficient portfolios in a simple case study, which is derived from the first one quoted in my paper.[1] He shows how the particular preferences of an investor can be expressed and used to select particular portfolios from the range of efficient portfolios. He also shows that in the case study which is used for illustration the portfolio which emerges from my approach to matching is not efficient, in the sense of portfolio theory.

The two-security case

2.5 My objective in this paper is to examine the relationship between the matching portfolio and the range of efficient portfolios. I shall concentrate on the same elementary case study, which Wilkie[(2)] introduced in sections 5 to 20. I shall follow Wilkie's notation, and the reader will need to refer to those sections because their contents are not repeated here.

2.6 A little algebra and arithmetic will enable us to investigate the selection of different portfolios using various alternative criteria. In the two-security problem the numbers x_1 and x_2 (which are the nominal holdings in securities S_1 and S_2 respectively) govern the expected surplus E, the variance of expected surplus V and the price of the portfolio P through the following three equations.

$$E = x_1E_1 + x_2E_2 - E_L$$
$$V = x_1^2V_1 + 2x_1x_2C_{12} + x_2^2V_2 - 2x_1C_{1L} - 2x_2C_{2L} + V_L$$
$$P = x_1P_1 + x_2P_2$$

2.7 Since there are only two distinct securities in our elementary case study, their amounts x_1 and x_2 would be determined from these equations if the investor were to specify his preferences in terms of values of E and P. (Alternatively he could specify either E and V or P and V.) However as Wilkie points out in sections 39 to 43 of his paper, the investor could alternatively specify his preferences in terms of gradients such as:

$$\lambda = \frac{\partial E}{\partial P} \quad \text{and} \quad \mu = \frac{\partial E}{\partial V}$$

2.8 These gradients represent the marginal trade-offs between the expected return and the price and the risk of the portfolio. As shown in Appendix 1, the following equations give the solution for the particular efficient portfolio which meets the investor's objectives in terms of these two parameters:

$$x_1 = \frac{\frac{1}{2\mu}(E_1V_2 - E_2C_{12}) - \frac{\lambda}{2\mu}(P_1V_2 - P_2C_{12}) + (C_{1L}V_2 - C_{2L}C_{12})}{V_1V_2 - C_{12}^2}$$

$$x_2 = \frac{\frac{1}{2\mu}(E_2V_1 - E_1C_{12}) - \frac{\lambda}{2\mu}(P_2V_1 - P_1C_{12}) + (C_{2L}V_1 - C_{1L}C_{12})}{V_1V_2 - C_{12}^2}$$

2.9 To follow Wilkie's worked example we may substitute the values which he gives in section 10 to arrive at:

$$x_1 = \frac{1}{\mu}(-1205{\cdot}35 - 2{\cdot}88669\,\lambda P_1 + 12{\cdot}6591\,\lambda P_2)$$
$$x_2 = \frac{1}{\mu}(6390{\cdot}83 + 12{\cdot}6591\,\lambda P_1 - 64{\cdot}5139\,\lambda P_2) + 10$$

2.10 For example if we put $P_1=400$, $P_2=100$, $\lambda=1{\cdot}2$, $\mu=4{,}800$ then we get straight to the first result quoted in Wilkie section 43, namely $x_1=-{\cdot}2233$ and $x_2=10{\cdot}9844$. It will be noted that the result is quite close to the minimum variance portfolio ($x_1=0$, $x_2=10$; see Wilkie section 28). The reason why this is so is that the parameter $\mu=\partial E/\partial V=4800$ represents an exceptionally risk averse investor: one who would be willing to invest in an alternative portfolio which increases his return by £48 but only if the variance of the surplus at the end of three years is increased by as little as £^{2}·01.

2.11 A typical investor would be less averse to risk than this and would normally accept a larger increase in the variance of surplus as the price to be paid for investing in an alternative portfolio which improves his expected return. The following table shows the solutions given in § 2.9 above for different values of λ and μ, together with the resulting values of P, E and V.

Table 1. *Prices* $P_1=400$, $P_2=100$

Parameters		Portfolio		Characteristics		
λ	μ	x_1	x_2	P	E	V
1·2	4800	− ·223	10·984	1009	994	·01
1·2	100	− 10·719	57·253	1438	5406	19·9
1·2	4·8	−223	994	10200	94760	8616
1·4	4800	− ·219	10·927	1005	987	·01
1·4	100	− 10·496	54·480	1250	5093	19·2
1·4	4·8	−219	937	6100	88245	8370

2.12 These examples show that when the lower values of μ are specified, i.e. $\mu=100$ or 4·8, then the solution is even more heavily weighted in favour of security 2 because of its much more favourable price than that of security 1. (The amount of security 1 is negative but for the present we shall not be concerned with restricting attention to the class of portfolios which do not contain negative asset holdings.)

2.13 The extreme price differential between the two securities was chosen by Wilkie in order to illustrate his points vividly. The following table repeats the one above but using more realistic prices $P_1=93$, $P_2=95$.

Table 2. *Prices* $P_1=93$, $P_2=95$

Parameters		Portfolio		Characteristics		
λ	μ	x_1	x_2	P	E	V
1·2	4800	− ·018	10·094	957	909	·00
1·2	100	− ·844	14·490	1298	1349	·16
1·2	4·8	−17·577	103·542	8202	10260	68·45
1·3	100	·090	9·538	914	854	·00
1·3	4·8	1·885	·385	212	− 53	·71
1·3	1·0	9·048	− 36·151	− 2593	− 3673	16·55
1·4	100	1·025	− 4·587	531	359	·23
1·4	4·8	21·346	−102·772	− 7778	−10366	99·27
1·4	1·0	102·463	−531·304	−40945	−53176	2289

2.14 Recalling that the liabilities amount to payments of 100 a year for three years, for which a fund of 300 should be more than adequate, some odd-looking portfolios result from this analysis. Neither the investment manager nor the actuary would be likely to find these particular portfolios well suited for their practical purposes. The next step therefore is to consider what parameters the investment manager and the actuary might like to specify in order to derive suitable portfolios.

3. PARAMETERS FOR INVESTMENT SELECTION

3.1 Let us consider the parameters which might be suitable for the investment manager and the actuary separately and in turn, as there is no reason to expect that the same parameters will be equally suitable for both.

The investment manager

3.2 The investment manager is faced with a practical constraint upon the price P which he can pay for the liabilities—namely the amount of the total fund at his disposal at any moment of time. Wilkie refers in sections 62 and 63 to this constraint. If we have under consideration the total liabilities and the total assets in hand, then the price P is fixed at that time and, as Wilkie says, the situation reduces to the conventional portfolio selection problem where the investor establishes his portfolio in terms of the balance between risk and return, i.e. between V and E. He can express his preference for risk V relative to return E in terms of the gradient $\mathrm{d}V/\mathrm{d}E$.

3.3 The conventional problem may be summarized in the following terms: analysis in terms of V and E gives a two-dimensional representation of all possible portfolios, the requirement that a portfolio should be 'efficient' (i.e. minimum V for given E) reduces the range of feasible portfolios to a one dimensional set, and the specification of $\mathrm{d}V/\mathrm{d}E$ as a further condition then yields a unique portfolio. In the three dimensional P–E–V analysis, the requirement of minimum V for given P and E reduces the range of portfolios under consideration to a two-dimensional set. In this context the investment manager's parameters, as specified in the preceding paragraph, are P and

$$\frac{\partial V}{\partial E}\left(=\frac{1}{\mu}\right).$$

These two parameters again yield a unique portfolio.

3.4 We can check the result of specifying parameters P and $1/\mu$ by re-formulating the solution given in § 2.8. The algebra is shown in Appendix 2 and the solution for the two-security case is given by the following two equations:

$$x_1 = \frac{P_2(P_2C_{1L}-P_1C_{2L}) + P(P_1V_2-P_2C_{12}) + \dfrac{1}{2\mu}P_2(E_1P_2-E_2P_1)}{P_1^2V_2-2P_1P_2C_{12}+P_2^2V_1}$$

$$x_2 = \frac{P_1(P_1C_{2L}-P_2C_{1L}) + P(P_2V_1-P_1C_{12}) + \frac{1}{2\mu}P_1(E_2P_1-E_1P_2)}{P_1{}^2V_2-2P_1P_2C_{12}+P_2{}^2V_1}$$

3.5 As before, substitute the values given by Wilkie in section 10 taking for example prices $P_1=93$, $P_2=95$. Then:

$$x_1 = 2{\cdot}314 - {\cdot}002436\ P + {\cdot}41865 \times \frac{1}{\mu}$$

$$x_2 = -2{\cdot}265 + {\cdot}012911\ P - {\cdot}40983 \times \frac{1}{\mu}$$

3.6 The following table shows a few solutions based on different parameters P and $1/\mu$:

Table 3. *Prices* $P_1=93$, $P_2=95$

Parameters		Portfolio		Characteristics		
P	$1/\mu$	x_1	x_2	P	E	V
200	0	1·827	·317	200·0	−68·0	·73
200	1	2·246	− ·093	200·0	−67·7	·88
200	2	2·664	− ·502	200·0	−67·4	1·31
400	0	1·340	2·899	400·0	190·1	·39
400	1	1·758	2·489	400·0	190·3	·54
400	2	2·177	2·080	400·0	190·7	·97
1000	0	−·122	10·646	1000·0	964·6	·00
1000	1	·297	10·236	1000·0	964·9	·15
1000	2	·715	9·826	1000·0	965·1	·58

3.7 Compared with the portfolios in Table 2 these results look reasonable and not unduly sensitive to the parameters. With the possible exception of the portfolios which involve negative asset holdings, any one of these might be suitable for the investment manager who has a particular size of fund (P) to invest relative to the given liabilities.

The actuary

3.8 The actuary responsible for assessing the ability of the fund to meet its liabilities is in a different position from that of the investment manager. For him, price is not a constraint at all; it is the objective of his calculations, which he will describe as the 'value of the liabilities'. The actuary needs to determine a suitable portfolio for the purposes of his valuation, but he will not necessarily regard either the portfolio actually being held or the investment manager's own preferred portfolio as ideal as a basis for the actuarial valuation. (Likewise the investment manager will not necessarily aim to hold the actuary's preferred portfolio.) The fundamental constraint upon the actuary is the expected eventual surplus E. If this figure differs greatly from zero, as it does in the examples tabulated above, then the value of P cannot reasonably be interpreted as the amount of assets required to meet the liabilities.

3.9 So the actuary ought to specify E. If he does, what are the implications? Whilst the investment manager fixes P and strikes a balance between E and V, the actuary would similarly fix E and seek to find a balance between P and V. For the actuary who has specified his main objective in terms of fixed E, the test of an efficient portfolio will be that it cannot be bettered by another portfolio with the same P and lower V or by one with the same V but lower P. The P–V criterion for the actuary is exactly analogous to the E–V criterion for the investment manager, the only difference being that the roles of E and P have been exchanged.

3.10 I suggest therefore that parameters for portfolio selection which the actuary would find suitable for the purposes of valuing the liabilities are E and $\partial V/\partial P$. These parameters are analogous to those of the investment manager discussed above, namely P and $\partial V/\partial E$, with the roles of E and P exchanged.

3.11 In this paper I have been referring to the variance V of ultimate surplus as the 'risk' of the portfolio. I suggest that we call the parameter $v = -\partial V/\partial P$ the 'degree of risk'. Providing that the parameter is positive, so that the gradient $\partial V/\partial P$ is negative, it represents the marginal trade-off between increased risk V and reduced price P which the actuary considers to be appropriate for his purposes. At the value $v=0$ the variance V of ultimate surplus attains its minimum for the chosen value of E, so the resulting portfolio can be described as the minimum risk portfolio relative to the liabilities for that particular value E of expected surplus. Values of v greater than zero will produce larger values of V for the same E. Therefore v is a measure of risk, as is V, but it measures the extent of departure from the particular portfolio at which V attains its minimum.

3.13 Let us therefore re-examine the two-security case using parameters E and v. Again we need to re-formulate the solution given in § 2.8. The algebra is shown in Appendix 3 and the solution is given by the following two equations:

$$x_1 = \frac{E_2(E_2\,C_{1L} - E_1\,C_{2L}) + (E + E_L)(E_1 V_2 - E_2 C_{12}) - \tfrac{1}{2}v\,E_2(P_1 E_2 - P_2 E_1)}{E_1^{\,2} V_2 - 2E_1 E_2\,C_{12} + E_2^{\,2} V_1}$$

$$x_2 = \frac{E_1(E_1\,C_{2L} - E_2\,C_{1L}) + (E + E_L)(E_2 V_1 - E_1 C_{12}) - \tfrac{1}{2}v\,E_1(P_2 E_1 - P_1 E_2)}{E_1^{\,2} V_2 - 2E_1 E_2\,C_{12} + E_2^{\,2} V_1}$$

It may be noted that this solution for the actuary's portfolio is analogous to that for the investment manager, as shown in § 3.4. The roles of E and P have been exchanged throughout and a term in E_L has now been introduced.

3.14 If we substitute the values given by Wilkie in section 10 we can write the solution thus:

$$x_1 = 1{\cdot}698 - {\cdot}001886\,E - ({\cdot}61287\,P_1 - {\cdot}60339 P_2)\,v$$

$$x_2 = {\cdot}998 + {\cdot}010002\,E + ({\cdot}60339\,P_1 - {\cdot}59405 P_2)\,v$$

If the prices are $P_1 = 93$, $P_2 = 95$, then:

$$x_1 = 1{\cdot}698 - {\cdot}001886\ E + {\cdot}32514\ v$$

$$x_2 = {\cdot}998 + {\cdot}010002\ E - {\cdot}31948\ v$$

3.15 Some examples for different parameters E and v are shown below:

Table 4. *Prices* $P_1 = 93$, $P_2 = 95$

Parameters		Portfolio		Characteristics		
E	v	x_1	x_2	P	E	V
0	0	1·698	·998	252·7	·0	·63
0	1	2·023	·679	252·6	·0	·72
0	2	2·348	·359	252·5	·0	·98
200	0	1·321	2·998	407·7	200·0	·38
200	1	1·646	2·679	407·6	200·0	·47
200	2	1·971	2·359	407·4	200·0	·73
400	0	·943	4·999	562·6	400·0	·20
400	1	1·269	4·679	562·5	400·0	·28
400	2	1·594	4·360	562·4	400·0	·54

3.16 The following table shows a few more results when the price differential between the two securities is more marked, namely when $P_1 = 90$ and $P_2 = 100$. The solution is:

$$x_1 = 1{\cdot}698 - {\cdot}001886\ E + 5{\cdot}1807\ v$$

$$x_2 = {\cdot}998 + {\cdot}010002\ E - 5{\cdot}0999\ v$$

Table 5. *Prices* $P_1 = 90$, $P_2 = 100$

Parameters		Portfolio		Characteristics		
E	v	x_1	x_2	P	E	V
0	0	1·698	·998	252·6	·0	·63
0	·5	4·289	−1·552	230·8	·0	6·11
0	1	6·879	−4·102	208·9	·0	22·54
200	0	1·321	2·998	418·7	200·0	·38
200	·5	3·911	·448	396·8	200·0	5·86
200	1	6·501	−2·101	375·0	200·0	22·28

3.17 The following points may be noted from Tables 4 and 5:

(1) When $v = 0$, the resulting portfolio for specified E is the same despite changes in the prices of the assets.
(2) When $v = 0$ and $E = 0$ the resulting portfolio is the same as the matching portfolio derived for the same worked example in section 3 of my paper.[(1)]
(3) As v is increased, the price P reduces and the variance V is increased, reflecting the cheaper but riskier portfolios which result.
(4) When the price differential is more marked, the results are more sensitive to the choice of v.

Furthermore it will be seen from the formula given in § 3.13 that the co-efficients of v vanish when $E_1/P_1 = E_2/P_2$.

Thus, when the expected yields on the two securities are the same, the chosen portfolio is independent of the value assigned to the degree of risk and is the same as the portfolio which is specified by $v=0$.

3.18 Portfolios with degree of risk equal to zero are clearly of interest. The particular portfolio which is specified by $E=v=0$ is, I believe, a natural starting point for the actuary who is concerned with the value to be placed on the liabilities. If the actuary wishes to ensure a probable residual surplus E as a margin in his valuation, he must shift to the portfolio specified by that value of E, still keeping $v=0$. If the yields on the two securities are identical then there is no need to specify v because the portfolio is selected by E alone. (This is the special case discussed by Wilkie in sections 65 to 69.) If the yields differ, the actuary can specify the degree of risk in terms of v and the portfolio will be shifted in favour of the higher-yielding stock. The value placed on the liabilities will thereby be reduced, but the uncertainty of the outcome of actually investing in such a portfolio will be increased because it is less well matched to the liabilities.

4. THE CRITERION OF EFFICIENCY

4.1 The portfolio specified by $E=0$, $v=0$, and the portfolio referred to in my earlier paper(1) as the 'unbiased match' are one and the same. The unbiased match was defined by the property that $E=0$ and the variance V (or equivalently the mean square ultimate surplus E^2+V) is minimized. Clearly if V is minimized for fixed E then:

$$v = -\frac{\partial V}{\partial P} = 0$$

The unbiased match therefore has parameters $E=0$, $v=0$.

4.2 It should be noted in passing that the (unconstrained) matching portfolio referred to in § 3.17 is not quite the same as the unbiased match. However these two definitions of matching for a stochastic model lead to very similar portfolios, and in our case study the portfolio $x_1 = 1{\cdot}698$, $x_2 = {\cdot}998$ is correct for both to the third decimal place.

4.3 Wilkie(2) pointed out, in sections 35 and 36, that the unbiased match for our case study is not efficient in the P–E–V sense when the expected yields on the two securities differ. For example, the matching portfolio can be improved upon by switching to another portfolio with the same V but larger E and smaller P. With the knowledge that such a switch can be made, the matching portfolio would appear to be one which should be disregarded by both the investment manager and the actuary.

4.4 However Wilkie elaborated upon the case study by demonstrating the following points:

(i) If the expected yields on the two securities are identical, the unbiased match is an efficient one. (See section 69.)
(ii) In conditions of realistic prices for the two securities, the unbiased match is almost indistinguishable from a portfolio which is efficient. (See section 74.)

It does not seem unreasonable to draw the general conclusion that it is only in conditions of unrealistic prices that the inefficiency of the matching portfolio is likely to become significant.

4.5 It appears therefore that, at least in our particular case study, the inefficiency of the matching portfolio cannot be of practical importance. Indeed it is not surprising, on reflection, to note that if there is a price advantage in favour of one of the securities then the composition of the matching portfolio—which is calculated without regard to the yields on the securities—can be improved upon for investment purposes.

4.6 The parameter v can be regarded as a measure of the departure from the matching portfolio. If the value $v=0$ generally yields an inefficient portfolio, it is of interest to consider at what value of v the selected portfolio becomes efficient. This point marks the frontier of the region of efficient portfolios for the given value of E. The solution is shown in Appendix 4. It depends upon the prices of the securities and the result is as follows for our case study:

$$\text{Frontier value of } v=\frac{900-E}{6391P_2-1205P_1}$$

4.7 Take for example the case $E=0$. Given eccentric prices $P_1=400$, $P_2=100$, the frontier value of v is ·0057. At more realistic prices $P_1=93$, $P_2=95$, the frontier value is ·0018. Above these values the selected actuary's portfolio will be efficient, whilst below these values (including $v=0$) the portfolio will be inefficient.

4.8 The frontier values are exceedingly small. A value of $v=-\partial V/\partial P=\cdot 0018$ means that the actuary considers it worth accepting an extra £1 of cost in order to invest in a lower risk portfolio which reduces the variance of ultimate surplus by as little as £^{2}.0018. This is a trifling reduction in the risk, as can be seen by looking at the values of V for the various portfolios shown in tables 4 and 5. In other words in this case study any reasonable degree of risk (meaning a value of v well above the frontier value) will lead the actuary to an efficient portfolio. This observation can hold good even when the prices are eccentric.

5. GENERALIZATION

5.1 The analysis may be generalized to any number n of securities, where n is equal to or greater than 2. I shall broadly follow the notation used by Wilkie in sections 75 to 78 except that the liabilities will not be included as the $(n+1)$th asset. Thus:

$x = (x_1, \ldots\ldots\ldots, x_n)$ is the n-vector which represents the nominal holding in each security;
$e = (E_1, \ldots\ldots, E_n)$ where E_i is the expected rolled-up return on security i (disregarding the liabilities);
$p = (P_1, \ldots\ldots, P_n)$ where P_i is the price per unit of security i;
$V = (V_{ij})$ the $n \times n$ matrix whose elements are the covariances between the rolled-up returns on security i and security j (the variances where $i = j$);
$c = (C_1, \ldots\ldots, C_n)$ where C_i is the covariance between the rolled-up returns on the liabilities and security i;
V_L is the variance of the rolled-up return on the liabilities; and E_L is the expected value of the rolled-up return on the liabilities.

5.2 The three characteristics of P, E and V can be expressed in the following manner:

$$P = x' p$$
$$E = x' e - E_L$$
$$V = x' V x - 2 x' c + V_L$$

If we require minimum V for any particular values of P and E we are left with a two-dimensional set of minimum variance portfolios. Two independent parameters, such as P and E themselves will specify a single portfolio. Let us specify the parameters E and v, as before.

5.3 The general solution for n securities is derived in Appendix 5. It is as follows:

$$x = x^\circ + E y + v z,$$

where:

$$x^\circ = V^{-1}(c + k_1 e)$$
$$y = k_2 V^{-1} e$$
$$z = \tfrac{1}{2} V^{-1}(k_3 e - p)$$

and

$$k_1 = \frac{E_L - e' V^{-1} c}{e' V^{-1} e}$$
$$k_2 = \frac{1}{e' V^{-1} e}$$
$$k_3 = \frac{e' V^{-1} p}{e' V^{-1} e}$$

5.4 This general formula is in the same format as the two security formula in §3.13, namely a linear function of E and v. The three additive terms within the linear formula have some striking properties which will be discussed in turn.

The constant term

5.5 The constant term x° is the unbiased match, as defined by the property that $E=0$ and V is minimized, i.e. $v=0$. As described in my earlier paper[1], this portfolio matches the liabilities by nature and by term. Thus the pattern of asset cash flows will tend to emulate the pattern of liability cash flows, fixed monetary liabilities will tend to be matched by fixed monetary assets, and inflation-linked or 'real' liabilities will tend to be matched by 'real' assets.

5.6 It is notable that the other vector terms $\boldsymbol{y}$ and $\boldsymbol{z}$ in the general solution are independent of the liabilities. The unbiased match can be said to represent the pure liability-related component of any efficient portfolio. For a slightly more picturesque description of this property we might say that all the information about the liabilities that is needed to select the portfolio $\boldsymbol{x}$ is conveyed within the unbiased match $\boldsymbol{x}^\circ$.

5.7 The unbiased match is independent of the prices of the securities and of their expected yields. For given liabilities it remains constant in times of changing market conditions, and it is also independent of the actuary's preferences relating to the degree of risk v and expected return E. Examples of matching portfolios were illustrated in my earlier paper.[1]

5.8 It may also be noted in passing, although the details will not be shown, that the general solution for $\boldsymbol{x}^\circ$ is algebraically equivalent to the solution for the unbiased match given as Theorem 3 of my theoretical paper on matching.[4]

The E component

5.9 The second component in the formula, namely $E\boldsymbol{y}$, is governed by the actuary's preference for expected return E but is independent of his preference, as measured by v, for risk. The vector $\boldsymbol{y}$ gives the direction in which the portfolio should be shifted away from the unbiased match in order to increase or decrease expected return without affecting the degree of risk. It is independent of the liabilities and it is also independent of the prices of the securities.

5.10 In practice if the actuary chooses to specify a margin E of ultimate surplus, the value of E which he chooses will no doubt be conditioned by the scale of the liabilities. If the liabilities are increased by a factor of 10 then presumably the margin would be increased likewise. Nevertheless for given E the second component of the formula for the selected portfolio, namely $E\boldsymbol{y}$, can be seen to represent the pure return-related component of the portfolio, independent of both the liabilities and the degree of risk in the portfolio.

The v component

5.11 The third component, $v\boldsymbol{z}$, is governed by the actuary's preference for risk as measured by $v=-\partial V/\partial P$. It is however independent of the return on the portfolio; it is easily verified that $\boldsymbol{e}'\boldsymbol{z}=0$ so that the entire contribution to the return on the portfolio results from the second component, $E\boldsymbol{y}$. The vector $\boldsymbol{z}$ gives the direction in which the portfolio should be shifted away from the unbiased match (or from $\boldsymbol{x}^\circ+E\boldsymbol{y}$) in order to increase or decrease the degree of risk on the portfolio without altering the expected return. It is independent of the liabilities.

5.12 The third component vz depends on the prices of the securities, whilst the other two components of the portfolio x do not. All the relevant information about the state of the investment market at the time of calculation is contained within this term.

5.13 If the state of the market is such that the expected yields E_i/P_i on all the securities are identical, then the third term is zero. This is easily verified: if v is the common rate of discount then $P_i = v\,E_i$ for all i, $\boldsymbol{p} = v\,\boldsymbol{e}$, $k_3 = v$ and so $\boldsymbol{z} = \boldsymbol{0}$. (See § 5.3.) It should be noted that the expected yield E_i/P_i relates to the period between now and the time horizon when all liabilities have expired; this yield will include the effects of re-investment of the proceeds of the security and will not in general be the same as its current market redemption yield.

5.14 If the prices of securities are such that their expected yields are not all the same, then k_3 can be interpreted as an average discount factor of the form:

$$k_3 = \frac{\Sigma\, w_i\, P_i}{\Sigma\, w_i\, E_i}$$

where the weights w_i are the elements of the vector $\boldsymbol{e}'V^{-1}$. (Equally the weights may be taken as the elements of $\boldsymbol{y}$.)

5.15 The vector $k_3\,\boldsymbol{e} - \mathbf{p}$ which determines z can be interpreted in the following way. Each element k_3E_i is the discounted present value of the rolled-up return on security S_i, where the rate of discount is the mean rate k_3. The element $k_3E_i - P_i$ is therefore a sort of present value, the size of which depends on the expected yield E_i/P_i on the security. In this way the vector z points towards or away from individual securities, or is indifferent towards them, according to their relative yields.

5.16 It is shown in Appendix 6 that the price of portfolio z is necessarily less than or equal to zero. Generally the price will be negative when z itself is non-zero. The price P of the selected portfolio $\boldsymbol{x}$, which in the terms of the actuary represents the value of the liabilities, is the sum of the prices of the three components $\boldsymbol{x}^\circ$, $E\boldsymbol{y}$ and $v\boldsymbol{z}$. Therefore if E is held constant, the effect of increasing v will be to reduce P. Furthermore, for values of v greater than zero: $\partial V/\partial P = -v$ is negative. Therefore as v is increased and P decreased for constant E, V must increase.

6. REVIEW OF THE CASE STUDY

6.1 It is worth reviewing our simple worked example in the light of the general analysis described above. Algebraic expressions for $\boldsymbol{x}^\circ$, $\boldsymbol{y}$ and $\boldsymbol{z}$ readily follow from the general solution of § 5.3 in the simple case $n = 2$. The workings are not shown here, but it can be verified that these three components of the portfolio $\boldsymbol{x}$ agree with the constants, the coefficients of E and the coefficients of v respectively in the two-security solution (see § 3.13).

6.2 Numerical solutions for the case study at prices $P_1 = 93$, $P_2 = 95$ were given in § 3.14. They may now be written in the following way:

$$x_1^\circ = 1{\cdot}698 \qquad x_2^\circ = {\cdot}998$$
$$y_1 = -{\cdot}001886 \qquad y_2 = {\cdot}010002$$
$$z_1 = {\cdot}32514 \qquad z_2 = -{\cdot}31948$$

What do these values mean or represent?

6.3 The values for $\boldsymbol{x}^\circ$ give the amounts of the two securities required to produce the best match, in the defined sense, of the asset cash flows to the liability cash flows. A change in the cash flows of the liabilities would change the values of $\boldsymbol{x}^\circ$ but not those of $\boldsymbol{y}$ or $\boldsymbol{z}$.

6.4 The values of $\boldsymbol{y}$ give the amounts of the two securities required to produce a change in the expected return without affecting the degree of risk. In this example an increase in expected return in such a manner would call for more investment in security S_2 and slightly less in security S_1. The reason why security S_2 is favoured for the purpose is evidently that it is the longer dated of the two and most of its proceeds emerge at the horizon date three years hence. The variance of the rolled up return on this security is relatively small, and so therefore is its effect on the variance of ultimate surplus net of the liabilities.

6.5 The values of $\boldsymbol{z}$ give the amounts of the two securities required to change the degree of risk without affecting the expected return. An increase in the degree of risk in such a manner would call for more investment in security S_1 and a broadly similar amount of disinvestment in S_2. The reason why S_1 is favoured for this purpose is that it has the higher expected yield. Thus:

$$\frac{E_1}{P_1} = \frac{120{\cdot}881}{93} = 1{\cdot}2998$$

$$\frac{E_2}{P_2} = \frac{122{\cdot}781}{95} = 1{\cdot}2924$$

6.6 Even such a slight difference between the expected yields is enough to have a significant effect on the chosen portfolio; as we have seen if the expected yields were identical then $\boldsymbol{z}$ would be zero.

7. SUMMARY

7.1 If we consider Wilkie's *P–E–V* portfolio selection model for the purposes of actuarial valuation, in the way that I have suggested, we are led to see any efficient portfolio as composed of three mutually exclusive parts:

(a) the unbiased match, which relates to the liabilities and is independent of the return and risk;
(b) the return-related part which is independent of the liabilities and of the degree of risk; and
(c) the risk-related part which is independent of the liabilities and of the return.

7.2 For the actuary whose objective is to value the liabilities, the answer may be taken as P, the price of the portfolio which is selected by his chosen parameters E and v. The price P is the sum of the prices of the above three components of the portfolio, of which only the first component depends on the liabilities. The total price of the risk and return-related components is therefore in the nature of an adjustment from the price of the unbiased match. The price adjustment is the same, for given E and v, whatever the liabilities being valued.

7.3 In my previous paper[1] I dealt with the problem of restricting attention to matching portfolios which contain no negative asset holdings. I suggested that the resulting class of 'positive match' portfolios had practical application in the field of actuarial valuation, and I quoted a variety of results. In the discussion on that paper, speakers rightly pointed out that actuarial valuations based on pure matching did not take account of the likely financial advantage to the fund of favouring more risky but potentially more profitable investments. This is the point which has now been addressed: the actuary can choose his preferred values for the parameters E and v which govern return and degree of risk in the portfolio relative to the liabilities. I am most grateful to David Wilkie for evolving and for explaining so clearly in his paper a generalized framework of portfolio theory within which such concepts can be developed.

7.4 The logical next step is therefore to investigate the problem of finding optimum portfolios with prescribed values of E and v but with no negative asset holdings. It will then be possible to re-work some of the examples quoted in my earlier paper in the new framework. At the time of writing, the necessary further groundwork has been prepared and I have obtained results which look most encouraging. I propose to publish this further study in a subsequent part of the *Journal*.

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APPENDIX 1

TWO SECURITY PROBLEM—SOLUTION IN TERMS OF λ AND μ

When the choice of portfolio is restricted to amounts x_1 and x_2 of the two specified securities, then E, V and P are all functions of x_1 and x_2. Furthermore as shown by Wilkie (section 23) V is a quadratic function of E and P. Therefore E can be expressed (by 'solving' the quadratic) as a function of V and P.

The following differential equation will hold:

$$\frac{\partial E}{\partial x_1} = \frac{\partial E}{\partial P}\frac{\partial P}{\partial x_1} + \frac{\partial E}{\partial V}\frac{\partial V}{\partial x_1}$$

$$= \lambda \frac{\partial P}{\partial x_1} + \mu \frac{\partial V}{\partial x_1}$$

where $$\lambda = \frac{\partial E}{\partial P}$$

and $$\mu = \frac{\partial E}{\partial V}$$

Now,

$$\frac{\partial E}{\partial x_1} = E_1, \frac{\partial P}{\partial x_1} = P_1 \text{ and } \frac{\partial V}{\partial x_1} = 2x_1 V_1 + 2x_2 C_{12} - 2C_{1L}$$

so

$$E_1 = \lambda\, P_1 + 2\mu(x_1 V_1 + x_2 C_{12} - C_{1L})$$

The corresponding equation for $\partial E/\partial x_2$ yields the similar result

$$E_2 = \lambda\, P_2 + 2\mu(x_2 V_2 + x_1 C_{12} - C_{2L})$$

These two equations for E_1 and E_2 can be solved for x_1 and x_2 and the result is as shown in §2.8.

APPENDIX 2

TWO SECURITY PROBLEM—SOLUTION IN TERMS OF P AND $1/\mu$

Write $v = -\frac{\partial V}{\partial P}$, and note that $\frac{1}{\mu} = \frac{\partial V}{\partial E}$

As mentioned in Appendix 1, V is a function of P and E.

Therefore
$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial P}\frac{\partial P}{\partial x_1} + \frac{\partial V}{\partial E}\frac{\partial E}{\partial x_1}$$

i.e.
$$2x_1V_1 + 2x_2C_{12} - 2C_{1L} = -vP_1 + \frac{1}{\mu}E_1 \tag{1}$$

Writing $x_3 = \frac{1}{2}v$:

$$x_1V_1 + x_2C_{12} + x_3P_1 = C_{1L} + \frac{1}{2\mu}E_1 \tag{2}$$

The corresponding equation for $\partial V/\partial x_2$ yields:

$$x_1C_{12} + x_2V_2 + x_3P_2 = C_{2L} + \frac{1}{2\mu}E_2 \tag{3}$$

Also
$$x_1P_1 + x_2P_2 = P \tag{4}$$

Equations (2), (3) and (4) may be solved for x_1. To do so multiply the first by P_2^2, the second by $-P_1P_2$, the third by $P_1V_2 - P_2C_{12}$ and add all three together. The result is:

$$x_1(P_2^2V_1 - 2P_1P_2C_{12} + P_1^2V_2)$$
$$= P_2^2(C_{1L} + \frac{1}{2\mu}E_1) - P_1P_2(C_{2L} + \frac{1}{2\mu}E_2) + P(P_1V_2 - P_2C_{12})$$

The solution for x_2 is obtained by multiplying the first of the above three equations by $-P_1P_2$, the second by P_1^2, the third by $P_2V_1 - P_1C_{12}$ and adding all three. The result is:

$$x_2(P_1^2V_2 - 2P_1P_2C_{12} + P_2^2V_1)$$
$$= P_1^2(C_{2L} + \frac{1}{2\mu}E_2) - P_1P_2(C_{1L} + \frac{1}{2\mu}E_1) + P(P_2V_1 - P_1C_{12})$$

These solutions for x_1 and x_2 are the same, after rearranging terms, as those shown in § 3.4.

APPENDIX 3

TWO SECURITY PROBLEM—SOLUTION IN TERMS OF E AND v

Starting with equation (1) of Appendix 2, re-define:

$$x_3 = -\frac{1}{2\mu}$$

$$x_1 V_1 + x_2 C_{12} + x_3 E_1 = C_{1L} - \tfrac{1}{2} v P_1 \qquad (5)$$

The corresponding equation for $\partial V / \partial x_2$ yields:

$$x_1 C_{12} + x_2 V_2 + x_3 E_2 = C_{2L} - \tfrac{1}{2} v P_2 \qquad (6)$$

Also $$x_1 E_1 + x_2 E_2 = E + E_L \qquad (7)$$

To solve for x_1, multiply the first of these three equations by $E_2{}^2$, the second by $-E_1 E_2$, the third by $E_1 V_2 - E_2 C_{12}$ and add all three together. The result is:

$$x_1(E_2{}^2 V_1 - 2E_1 E_2 C_{12} + E_1{}^2 V_2)$$
$$= E_2{}^2(C_{1L} - \tfrac{1}{2} v P_1) - E_1 E_2 (C_{2L} - \tfrac{1}{2} v P_2) + (E + E_L)(E_1 V_2 - E_2 C_{12})$$

To solve for x_2, multiply the first of the three equations by $-E_1 E_2$, the second by $E_1{}^2$, the third by $E_2 V_1 - E_1 C_{12}$ and add all three. The result is:

$$x_2(E_1{}^2 V_2 - 2E_1 E_2 C_{12} + E_2{}^2 V_1)$$
$$= E_1{}^2(C_{2L} - \tfrac{1}{2} v P_2) - E_1 E_2 (C_{1L} - \tfrac{1}{2} v P_1) + (E + E_L)(E_2 V_1 - E_1 C_{12})$$

These formulae for x_1 and x_2 are equivalent to those shown in § 3.13.

APPENDIX 4

THE VALUE OF v AT THE EFFICIENT FRONTIER

Wilkie describes the regions of efficient portfolios in section 33. Providing we are concerned with values of E which are not unduly large for the purpose of actuarial valuation (to be precise values of E no greater than 900 in the case study), the dividing line (strictly speaking plane) between efficient and inefficient portfolios is given by the condition $bE+hP+f=0$.

An equivalent condition is $\partial V/\partial E=0$. (See Wilkie section 24 for the general form of V, from which this equivalence can be seen.) We can find the value of v at which $\partial V/\partial E=0$ by setting $x_3=0$ in the three equations (5), (6), and (7) of Appendix 3. The equations are simplified to the following form:

$$x_1V_1 + x_2C_{12} + \tfrac{1}{2}vP_1 = C_{1L}$$
$$x_1C_{12} + x_2V_2 + \tfrac{1}{2}vP_2 = C_2L$$
$$x_1E_1 + x_2E_2 = E + E_L$$

Multiply the first equation by $E_2C_{12}-E_1V_2$, the second by $E_1C_{12}-E_2V_1$, the third by $V_1V_2-C_{12}^2$ and add the three results together. We arrive at the following solution.

$$v=\frac{E_c-E}{n_1P_1+n_2P_2}$$

where: E_c is the value of E for the minimum variance portfolio (see Wilkie section 28)

$$n_1=\frac{\tfrac{1}{2}(E_1V_2-E_2C_{12})}{V_1V_2-C_{12}^2}$$

$$n_2=\frac{\tfrac{1}{2}(E_2V_1-E_1C_{12})}{V_1V_2-C_{12}^2}$$

APPENDIX 5

GENERAL SOLUTION IN TERMS OF E AND v

In the general n security problem the equations for V, P and E are as shown in §5.2.

For minimum variance portfolios, V is a function of P and E. (See Wilkie section 77.) Therefore, for security 1:

$$\frac{\partial V}{\partial x_1}=\frac{\partial V}{\partial P}\frac{\partial P}{\partial x_1}+\frac{\partial V}{\partial E}\frac{\partial E}{\partial x_1}$$

i.e. $2(\boldsymbol{Vx})_1-2C_1=-vP_1+\frac{1}{\mu}E_1$

There are n equations of this form for the n securities, and they may be expressed simultaneously in vector form thus:

$$2\,\boldsymbol{Vx}-2\boldsymbol{c}=-v\boldsymbol{p}+\frac{1}{\mu}\boldsymbol{e}$$

Assuming that V can be inverted (which will not be investigated here):

$$\boldsymbol{x}=\boldsymbol{V}^{-1}(\boldsymbol{c}+\frac{1}{2\mu}\boldsymbol{e}-\tfrac{1}{2}v\boldsymbol{p}) \tag{8}$$

Also: $E+E_L=\boldsymbol{e}'\boldsymbol{x}$

So $E+E_L=\gamma+\frac{1}{2\mu}\alpha-\frac{1}{2}v\beta$

where
$$\alpha=\boldsymbol{e}'\boldsymbol{V}^{-1}\boldsymbol{e}$$
$$\beta=\boldsymbol{e}'\boldsymbol{V}^{-1}\boldsymbol{p}$$
$$\gamma=\boldsymbol{e}'\boldsymbol{V}^{-1}\boldsymbol{c}$$

Therefore $\frac{1}{2\mu}=\left[\frac{E_L-\gamma}{\alpha}\right]+\frac{E}{\alpha}+\frac{1}{2}v\frac{\beta}{\alpha}$

Substituting in equation (8) for x gives the solution shown in §5.3:

$$\boldsymbol{x}=\boldsymbol{V}^{-1}\left(\boldsymbol{c}+\left[\frac{E_L-\gamma}{\alpha}\right]\boldsymbol{e}\right)+\frac{E}{\alpha}\boldsymbol{V}^{-1}\boldsymbol{e}+\tfrac{1}{2}v\boldsymbol{V}^{-1}\left(\frac{\beta}{\alpha}\boldsymbol{e}-\boldsymbol{p}\right)$$

APPENDIX 6

PROOF THAT THE PRICE OF z IS NEGATIVE OR ZERO

The price of portfolio z is:

$$\begin{aligned} p'z &= \tfrac{1}{2} p' V^{-1}\left(\frac{\beta}{\alpha} e - p\right) \\ &= \frac{1}{2\alpha}(\beta^2 - \alpha\delta) \end{aligned}$$

where α and β are as defined in Appendix 5 and

$$\delta = p' V^{-1} p$$

Since V is a covariance matrix it is positive semi-definite;

$$\text{i.e. } x'Vx \geqslant 0 \qquad \text{for all } x.$$

V^{-1}, if it exists, is also positive semi-definite because

$$x'V^{-1}x = (x'V^{-1})V(x'V^{-1})' \geqslant 0.$$

By a standard theorem of linear algebra, V^{-1} can therefore be expressed in the form $V^{-1} = Q'Q$ where Q is n by n.

Write $\quad a = Qe \quad , \quad b = Qp$

Then

$$\begin{aligned} \alpha\delta - \beta^2 &= (a'a)(b'b) - (a'b)^2 \\ &= (\Sigma a_i^2)(\Sigma b_j^2) - (\Sigma a_i b_i)^2 \\ &= \sum_i \sum_j (a_i^2 b_j^2 - a_i b_i\, a_j b_j) \\ &= \sum_i \sum_j a_i b_j (a_i b_j - a_j b_i) \\ &= \sum_{i<j} (a_i b_j - a_j b_i)^2 \\ &\geqslant 0 \end{aligned}$$

Since $\quad \alpha > 0 \quad , \quad p'z \leqslant 0.$