Pricing maturity guarantees in a regime-switching diffusion market

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October 21, 2010

Abstract

We consider the pricing of maturity guarantees for insurance contracts in a regime-switching lognormal market model. Regime-switching models have been empirically shown to fit long-term stockmarket data better than many other models. As the market is incomplete, there is no unique price for a maturity guarantee. We extend the good-deal pricing bounds idea to the regime-switching lognormal market model. This allows us to obtain a reasonable range of prices for the maturity guarantee, by excluding those prices which imply a Sharpe Ratio which is too high. As an illustration, we calculate the good-deal pricing bounds for maturity guarantees of various maturities.

Contents

Chapter 1

Introduction

Maturity guarantees are a common addition to many life insurance policies. The policyholder is given a guarantee by the life insurance company that the proceeds of the policy at the maturity date is subject to a minimum value. Ensuring that the guarantee is properly valued is of concern to the life insurance company, since it is a potential threat to the solvency of the company. When investment market returns are depressed, the company's investments are reduced in value but this is precisely the time when the guarantee is likely to bite. Thus the financial burden of the guarantee on the company is exacerbated.

Suppose that a policy is sold today for a single premium of £1000. The insurance company invests the premium in the stockmarket and, in 10 years time, it pays the proceeds to the policyholder, if he is still alive. In order to make the policy more attractive, the company guarantees that the amount paid to the policyholder at maturity will not be less than 75% of the premium. Thus the policyholder is guaranteed to receive at least £750 at the maturity date.

The inclusion of a minimum payout at maturity is called an *embedded option*. It is embedded into the insurance contract in the sense that it cannot be traded separately. There are many types of embedded options, such as surrender options, minimum return guarantees and annuity rate guarantees. All embedded options have an intrinsic value. Moreover, since insurance contracts are of a long-term nature, the economic conditions at the maturity date of a policy can be very different to those prevailing when the policy was issued. The implication is that an embedded option which may have had negligible worth at the outset of the policy can become very valuable by the time the policy matures.

The risks of embedded options must also be carefully accounted for, since the potential payout on a portfolio of policies with the same embedded option becomes larger when the number of policies in the portfolio increases. To illustrate this, suppose an insurance company sells 5000 single-premium policies which pay after 10 years the invested proceeds to any surviving policyholders. The risk that the insurance company bears is that more policyholders survive than expected. By the law of large numbers, this risk should decline as more policies are sold. Compare this to the insurance company selling 5000 of the above policies, but including a maturity guarantee of £750. The insurance company still bears the mortality risk. However, if after 10 years the proceeds are only £700 then the insurance company must pay each surviving policyholder an additional £50. This is a binary outcome, where either all the policies generate claims or none do. With the maturity guarantee, there is a risk factor which is common to all policies and, unlike mortality risk, it is not reduced by selling more of the same policies.

To begin to quantify the risks inherent in an embedded option, we must value them appropriately. The primary aim of this paper is to obtain a method for the reasonable valuation of maturity guarantees within a model which is appropriate for the long-term nature of the guarantees. To explain how we do this, we need to introduce a model of the market and some ideas from financial economics.

It is well-known that maturity guarantees have the same payoff as a European put option (for example, see Boyle and Hardy (1997)). To show this, denote the maturity date of an insurance contract by T and suppose that the guaranteed benefit is amount K at time T. If the amount payable before the minimum guarantee is applied is $S(T)$ at time T, then the policyholder receives $\max[K, S(T)]$ at time T. This means that the insurance company is liable to pay an additional amount of $K - S(T)$ to the policyholder if the guarantee bites at the maturity date. We can write this mathematically as

$$\max\left[K - S(T), 0\right].$$

The above cost to the insurer is recognised as the payoff of a *European put option* with strike price K and maturity date T. Thus valuing the maturity guarantee is equivalent to valuing a European put option.

To value the maturity guarantee, we use ideas from financial economics which require a model of the financial market. We assume a model of the stockmarket called a regime-switching lognormal ("RSLN") model. Regime-switching market models are a way of capturing discrete shifts in market behavior. These shifts could be due to a variety of reasons, such as changes in market regulations, government policies or investor sentiment. In particular, RSLN are effective at capturing the long-term behaviour of the stock market. This is an extremely appealing feature if we are valuing maturity guarantees since often the guarantees are applied after many years. First introduced by Hamilton (1989), regime-switching models have been shown in various empirical studies to be better at capturing market behavior than their non-regime-switching counterparts (for example, see Ang and Bakaert (2002), Gray (1996) and Klaassen (2002)).

An example of regime-switching market is one in which there are only two regimes: a bear market regime and a bull market regime. Suppose the market starts in a bull market regime, in which prices are generally rising. It stays in this regime for a random length of time before switching to a bear market regime, in which prices are generally falling. It then stays in the bear market for another random length of time before switching back to the bull market. This cycle continues *ad infinitum*. We explain more about RSLN models in Chapter 2.

Unlike the classical Black-Scholes model, the RSLN model is not complete, which means that not all payoffs can be replicated. This has immediate consequences for the valuation of any option in the model, in that there is no longer a unique price for it. Instead, there is a range of prices called the *no-arbitrage bounds* that spans all of the possible market prices. As these bounds are too wide to be practically useful, various suggestions have been made on how to price options in incomplete markets. They can be separated into two camps: the selection of either a single price, chosen according to some criteria, or a narrow subset of the no-arbitrage bounds.

At some point, we require a single price - for example, to quote a price for selling a contract with a maturity guarantee. However, we also recognise that our chosen price may not agree with the market price, even if the market model is correct. How do we know that our chosen price is reasonable compared to what the market is likely to choose? To try to answer this, we construct a range of reasonable prices, which are tighter than the no-arbitrage bounds. We deem our chosen price as reasonable if it lies in this range.

To construct the narrower range of prices, we use the *good-deal bound* idea. First proposed by Cochrane and Saá Requejo (2000), the good-deal bound idea is based on the Sharpe Ratio, which is the excess return on an investment per unit of risk. The essential idea is to exclude the option prices which are deemed unreasonable, in that they arise in markets in which the Sharpe Ratio is too high, meaning that the option price is "too good to be true".

The good-deal bound idea was streamlined and extended to jump-diffusion markets by Björk and Slinko (2006). However, as it has not yet been extended to regime-switching diffusion models, we must extend it to RSLN model, which we do in Chapter 3. This involves techniques from stochastic control theory but ultimately we obtain a partial integro-differential equation which can be evaluated on a computer.

There have other suggestions as to how to narrow the range of possible prices. In Bayraktar and Young (2008), Sharpe Ratios are also used to price options in incomplete markets. However, the perspective is that of an individual seller of one option, rather than that of the entire market. The seller of an option decides the option price via his own risk preferences, as expressed by his own chosen Sharpe Ratio. In other words, the seller of the option chooses the risk-neutral martingale measure under which he prices the option. It is shown in Bayraktar and Young (2008) that the upper and lower good-deal bounds of Cochrane and Saá Requejo (2000) can be obtained; in that case, the seller's chosen risk-neutral martingale measure coincides with the martingale measure which gives the upper good-deal bound. The lower good-deal bound is obtained in Bayraktar and Young (2008) by considering the buyer of the option.

A utility-based approach to the good-deal bound idea is found in Černý (2003), and extended in Klöppel and Schweizer (2007). An alternative approach based on the gain-loss ratio, which is the expectation of an asset's positive excess payoffs divided by the expectation of its negative excess payoffs, is found in Bernardo and Ledoit (2000).

In summary, the aim of the paper is to obtain a range of reasonable values for maturity guarantees within a RSLN market model by using the good-deal bound idea. For simplicity, we ignore mortality and focus on the financial aspect of the valuation.

Chapter 2

The market model

We introduce a market model in which there is one stock and a risk-free asset. An example of a risk-free asset is a bank account and typical examples of stocks are equities, bonds or a pooled fund. To provide a suitable comparison, we begin by introducing the classical Black-Scholes model for the stock price process, before describing a regime-switching model.

We assume that all the processes introduced below are defined on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As we are only interested in finite time horizons, we consider only the time interval $[0, T]$, for some fixed $T \in (0, \infty)$.

2.1 The classical Black-Scholes model

The classical Black-Scholes model is a standard model to model stock returns, in which the stock price follows a geometric Brownian motion. Denoting the stock price at time t by $S(t)$, then the classical Black-Scholes assumes that

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t). \tag{2.1.1}$$

Here, W is a 1-dimensional standard Brownian motion. The market parameter μ is the *mean rate of return* and σ is the *volatility process* of the stock price. The stock price return up to time t is lognormally distributed, with mean μt and variance $\sigma^2 t$, that is

$$\ln \frac{S(t)}{S(0)} \sim N(\mu t, \sigma^2 t).$$

Due to this lognormal distribution, we describe a classical Black-Scholes model as a lognormal ("LN") model. If the classical Black-Scholes model has market parameters μ and σ, then compactly we denote it by $\mathrm{LN}(\mu, \sigma)$. A realisation for a stock with parameters $\mu = 0.12$ and $\sigma = 0.15$ is shown in Figure 2.1.

The classical Black-Scholes model is popular for a few reasons. It captures the small-scale random fluctations observed in real stock market data. It is also quite simple and tractable. The parameters μ and σ can be estimated using maximum likelihood estimation; the estimates are the mean and variance of the log returns. These are all very attractive features for a model.

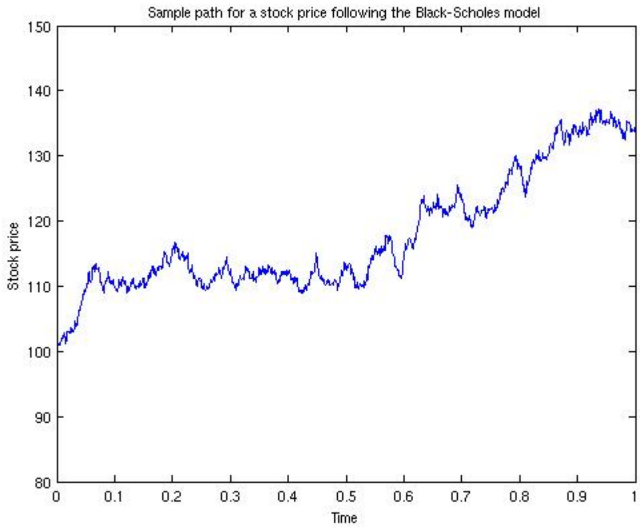


Figure 2.1: Sample path of a stock price which follows the Black-Scholes model.

The main drawback is that it fails to capture extreme price movements. Empirical studies of actual stock price movements show that extreme price movements are more common than the classical Black-Scholes model suggests. This means that the probability of observing very large price movements, whether positive or negative, is small in the classical Black-Scholes model. Another drawback is that it doesn't capture the empirically observed phenomenon of volatility clustering, where there are periods of high volatility followed by periods of low volatility.

2.2 The regime-switching lognormal model

In a regime-switching market model, the market switches between a fixed number of different regimes. Within each regime, the market is in a certain state, for example a state in which prices are generally rising, or in which the price volatility is high. Regime-switching models were first introduced by Hamilton (1989). In that paper, the market followed an autoregressive ("AR") model within regimes. Hamilton and Susmel (1994) studied regime-switching models where the market follows an autoregressive conditional heteroskedasticity ("ARCH") model within each regime.

We focus on a simple regime-switching market model called the regime-switching lognormal ("RSLN") model. In the RSLN model, the market follows an LN model within each regime. If the RSLN model has K regimes then we denote it by RSLN(K).

In order to have a mathematical description of the RSLN model, we define the process which drives the regime-switching. Denote by $\alpha(t)$ the regime that the market is in at time t. We assume that the process α is a *Markov chain*.

Definition 2.2.1. A Markov chain is a process $\alpha = \{\alpha(t); t \in [0, T]\}$ defined on a countable set $\mathcal{I}$ which satisfies the *Markov property*

$$\mathbb{P}\left[\alpha(t_n) = j \,|\, \alpha(t_1) = i_1, \ldots, \alpha(t_{n-1}) = i_{n-1}\right] = \mathbb{P}\left[\alpha(t_n) = j \,|\, \alpha(t_{n-1}) = i_{n-1}\right],$$

for all $j, i_1, \ldots, i_{n-1} \in \mathcal{I}$ and any sequence $t_1 < t_2 < \cdots < t_n$ of times.

In the RSLN model, the price process $S = \{S(t), t \in [0,T]\}$ of the stock satisfies

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(\alpha(t_-))\,\mathrm{d}t + \sigma(\alpha(t_-))\,\mathrm{d}W(t), \quad \forall t \in [0,T], \tag{2.2.1}$$

with the initial value $S(0)$ being a fixed, strictly positive constant. The parameters $\mu(i)$ and $\sigma(i)$ are constants and we assume further that $\sigma(i)$ is non-zero for each $i \in \mathcal{I}$. We assume that the Markov chain starts in a fixed state $i_0 \in \mathcal{I}$, so that $\alpha(0) = i_0$, almost surely.

Thus when $\alpha(t_-) = 1$ then the mean rate of return is $\mu(1)$ and the volatility is $\sigma(1)$. The use of $\alpha(t_-)$ rather than $\alpha(t)$ ensures that the market parameters are predictable, which is a technical condition.

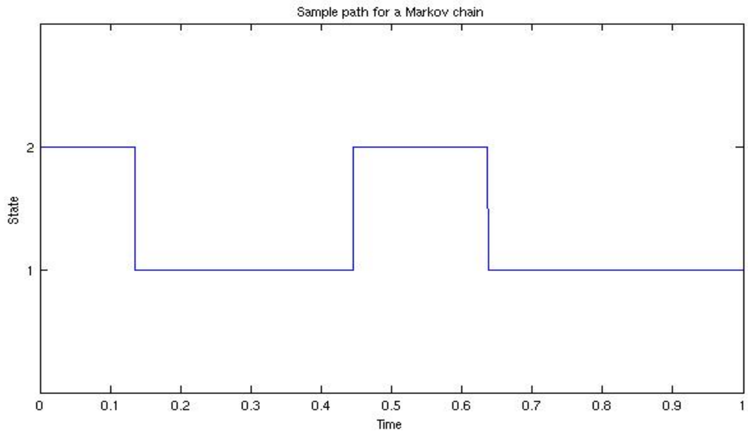
In order to complete the RSLN model, we need to specify the rates at which the Markov chain switches between states, or regimes. In continuous time, the evolution of the Markov chain is described by a matrix G called the *generator* of the chain. The generator is a $D \times D$ matrix $G = (g_{ij})_{i,j=1}^{D}$ with the properties

$$g_{ij} \geq 0, \quad \forall j \neq i \qquad \text{and} \qquad g_{ii} = -\sum_{j \neq i} g_{ij}.$$

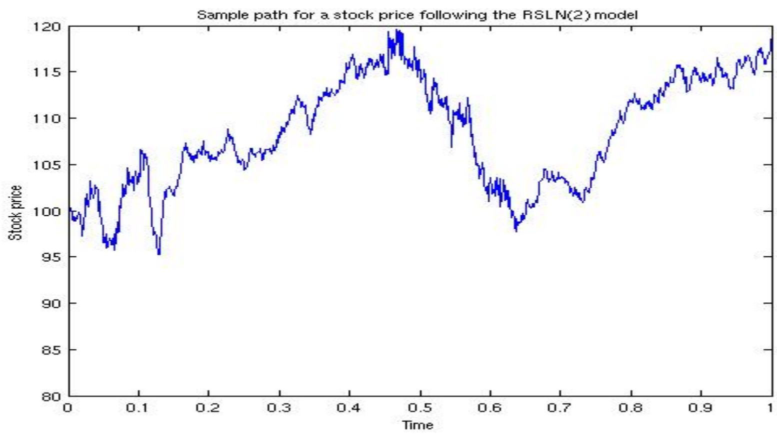
The interpretation of the off-diagonal element g_{ij} of the generator is as the instantaneous rate of transition from state i to state j.

The stock price dynamics (2.2.1) look very similar to the dynamics (2.1.1) in the classical Black-Scholes model except that the parameters μ and σ, instead of being constants, are now functions of the Markov chain $\alpha(t)$. This has a visible effect on the typical stock price dynamics in the RSLN model.

As an example, consider Figure 2.2 which illustrates a realisation from an RSLN(2) model. In this example, regime 1 corresponds to a low volatility environment with a positive mean rate of return. Regime 2 corresponds to a high volatility environment with a negative mean rate of return. Figure 2.2(a) shows a possible path of the Markov chain α. We see for this particular path that there are three changes in market regime. Figure 2.2(b) shows a possible path of the stock price, corresponding to the sample path of the Markov chain. The initial regime is regime 2, and the Markov chain stays in this regime until about time $t = 0.12$. During this time, the stock price is distributed as $\mathrm{LN}(\mu(2), \sigma(2))$; notice the volatility of the stock price in Figure 2.2(b) up to about $t = 0.12$. Around time $t = 0.12$, the market switches to regime 1 and stays in this regime until about time $t = 0.45$. The stock price is distributed as $\mathrm{LN}(\mu(1), \sigma(1))$ during this time period. See how the volatility decreases and there is a clear upward trend in the stock price. Around $t = 0.45$, the market regime switches back to regime 2, and then the stock price is once more distributed as $\mathrm{LN}(\mu(2), \sigma(2))$. Here the stock price volatility is seen to increase and there is a clear downward trend.



(a) A sample path of a Markov chain.



(b) The corresponding sample path of a stock price which follows the RSLN(2) model.

Figure 2.2: Sample paths for an RSLN(2) model.

2.3 The RSLN model

The RSLN model allows us to overcome some of the drawbacks of the classical Black-Scholes model, namely that it fails to capture extreme price movements, while retaining some of the tractability. In Hardy (2003, Chapter 3 and page 226), statistical tests suggest that for data from the S&P 500, TSE 300 and FTSE All-Share Total Return Index over the years 1956-2001, an RSLN(2) model provides a better fit than a range of other models, including the LN, AR(1), ARCH, GARCH and regime-switching AR(1) model. This strongly suggests that for models of long-term stockmarket behaviour, an RSLN model should be considered.

The price of this better fit is the need to estimate more parameters. For example, in a RSLN(2) model, we need to estimate 6 parameters: $\mu(1)$, $\mu(2)$, $\sigma(1)$, $\sigma(2)$, g_{11} and g_{22}. Compare this with the LN(μ, σ) model, where we needed to estimate only 2 parameters. However, given the improved fit to the data, this is not unduly onerous.

We use the RSLN model to describe the price dynamics of the stock. This means that the stock price process S satisfies (2.2.1). We assume that there are D market regimes. The market-switching between market regimes is modelled by a Markov chain α which takes values in a finite state space $\mathcal{I} = \{1, \ldots, D\}$. and has generator $G = (g_{ij})_{i,j=1}^{D}$.

In the market model there is also a risk-free asset. Consistent with the stock price dynamics, we assume that the risk-free asset price process B satisfies

$$\frac{\mathrm{d}B(t)}{B(t)} = r(\alpha(t_-))\,\mathrm{d}t, \quad \forall t \in [0,T], \quad B(0) = 1.$$

We call r the *risk-free rate of return*. The above equation can also be solved explicitly to find

$$B(t) = \exp\left\{\int_0^t r(\alpha(s_-))\,\mathrm{d}s\right\},$$

for all $t \in [0,T]$.

Remark 2.3.1. It is straightforward to generalise the above market model to include a finite number N of risky assets. It is also straightforward to further generalise the market parameters r, μ and σ to be of the form

$$r(t) = r(t, S(t), \alpha(t_-)), \quad \mu(t) = \mu(t, S(t), \alpha(t_-)), \quad \sigma(t) = \sigma(t, S(t), \alpha(t_-)).$$

However, for the sake of clarity, it is preferable to keep to the model we have outlined above.

2.4 Summary

The aim of the paper is to price maturity guarantees within the framework of the RSLN model. The RSLN model provides a better fit to long-term financial market data and hence it should be considered as model for the pricing of maturity guarantees, which often apply after many years have elapsed since the policy was first purchased. However, as we see in the next chapter, the pricing method is neither as straightforward nor as standard as in the classical Black-Scholes model. This reflects the difficulty in pricing in incomplete markets (of which the RSLN model is one) and the current uncertainty in the literature in what is the best method of pricing in incomplete markets. On the latter point, there are several choices of methodology. We apply a technique, called the good-deal bound, to the pricing of maturity guarantees in the RSLN model.

Chapter 3

Pricing options

The approach that we use to price options is rooted in the theory of finance. Beginning with the seminal ideas on option pricing of Black, Merton and Scholes, the growth of financial derivatives has been paralleled by developments in the field of mathematical finance. The reason is that the economic ideas of Black, Merton and Scholes can be translated into a mathematical framework, using the tools of stochastic calculus and martingale theory. Mathematics allows the derivative market participants to price and hedge products.

Given a model of the market, the first requirement is that there is an absence of arbitrage strategies. An arbitrage strategy is a trading strategy which, starting from zero wealth, generates profits without any risk. From the absence of arbitrage, it follows that the value of a derivative is the value of a trading strategy that replicates the derivative's payoffs (called the *replicating portfolio*). This consequence must hold, for otherwise there is an arbitrage strategy. To see this, suppose the price of the derivative is greater than the price of the replicating portfolio. Then we can sell the derivative and buy the replicating portfolio, leaving us with a positive amount of cash attained without incurring any risk.

Therefore, valuing a derivative in an arbitrage-free model amounts to finding and valuing the replicating portfolio. However, it is not always possible to find a replicating portfolio. In such models, of which the RSLN model is one, we need to find alternative approaches to value a derivative.

We have based the introduction to the financial theory of option pricing in Sections 3.1-3.3 on material from Björk (2009) and Hunt and Kennedy (2004).

3.1 Incomplete markets

To illustrate the idea of an incomplete market, we use a very simple market model in discrete time. Suppose that we are given a market model with only one traded asset S, with price S_0 at time 0. At time 1, the market can be in one of two possible states: either state ω_1 with probability p or state ω_2 with probability $1 - p$. At time 1, the price of the asset is $S_1(\omega_1)$ in state ω_1 and $S_1(\omega_2)$ in state ω_2. This setup is shown in Figure 3.1.

We wish to price a derivative X which pays amount $X_1(\omega_i)$ at time 1 when the market is in state ω_i, for $i = 1, 2$. An example of a derivative is a European put option with strike price K and maturing at time 1. The European put

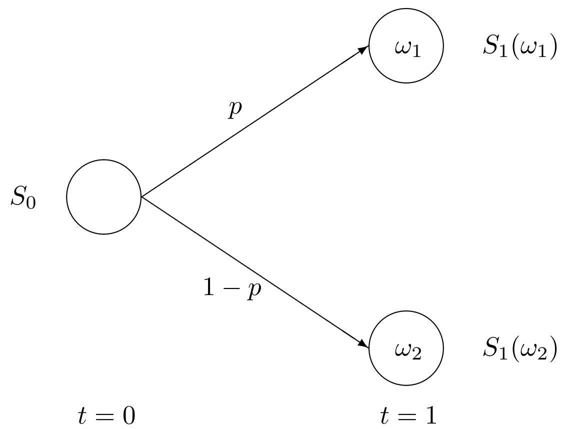


Figure 3.1: The possible states of the market and the possible prices of asset S.

option gives the buyer the option to sell the underlying stock at price K at time T. This means that at time 1,

- if $S_1 > K$ then the seller pays nothing to the buyer; and
- if $S_1 \leq K$ then the seller pays $K - S_1$ to the buyer.

The task is to calculate the time-0 price X_0 of the derivative. We do this by constructing a replicating portfolio. Denote the number of units held of asset S by ϕ. If we can find ϕ satisfying

$$\phi S_1(\omega_i) = X_1(\omega_i), \qquad \text{for } i = 1, 2, \tag{3.1.1}$$

then we call ϕ a *replicating portfolio* for the derivative X. Holding ϕ units of asset S at time 0 means that the time-1 payoff of the derivative is replicated, regardless of the state of the market. If such a portfolio ϕ exists then the fair price of the derivative at time 0 is the cost of the replicating portfolio, that is

$$X_0 = \phi S_0.$$

However, what if there is no replicating portfolio ϕ? Indeed, there is no reason why such a portfolio should exist in this simple model; we see from (3.1.1) that the one unknown ϕ must satisfy two equations. In that case, we say that the market is *incomplete* since there are payoffs (such as the time-1 payoffs of derivative X) that are not entirely determined by the prices of traded assets (for example, the price of asset S). This means that in an incomplete market there is an uncertainty about the value of these payoffs. In contrast, in a *complete* market all payoffs are entirely determined by the prices of traded assets and hence the values of all payoffs are known.

In real life, financial markets are incomplete. There are various possible sources of incompleteness, such as a lack of traded assets relative to the payoffs that an investor wishes to replicate; examples of these are temperature derivatives and catastrophe bonds. Market frictions, such as transaction costs and constraints on the investor's portfolio, can also cause incompleteness.

Given the reality, it appears more realistic to use an incomplete market model since this allows us to model the uncertainty arising from the incompleteness. However, while the theory of derivative pricing in complete markets is very well understood, there is still no sound, comprehensive framework for derivative pricing in incomplete markets. The problem in incomplete markets is that, applying the approach in complete markets to do pricing, there is no unique price for derivatives. Rather, a range of possible prices is obtained and various suggestions made on how to obtain either a single price or a narrower range of prices for each derivative.

When the incompleteness arises from a lack of traded assets, one suggestion is to make the market model into a complete one by a process called *fictional completion*. The idea is to introduce more assets into the model, and these new assets cannot be replicated by the existing assets. If we do this until there exists a replicating portfolio for every possible derivative of the assets (both existing and new) in the market, then we have *completed* the market. If we use the completed market model to price derivatives, then we obtain unique prices.

While the notion of completing the market is an attractive one, it does not solve the original problem. The assets that we use to complete the market are not traded in the market we seek to model, which means that we cannot observe their prices in the real world. The uniqueness of the derivative prices is based on the prices of these assets, and we are not certain what these asset prices should be since there is no objective way of determining them within the model. All we have done is hidden the uncertainty arising from the lack of traded assets in the prices of the assets used to complete the market, rather than expressing it openly in the prices of the derivatives.

There have been other suggestions made on derivative pricing in incomplete markets, but as they revolve around the notion of risk-neutral measures (as does our approach), we begin by explaining the latter concept.

3.2 Risk-neutral pricing

To explain risk-neutral pricing, we expand the simple market model above to include a second asset B. We assume that the time-0 price of B is B_0 and the time-1 price is $B_1(\omega_i)$, for $i = 1, 2$. This expanded market model is shown in Figure 3.2.

We wish again to price a derivative X which pays amount $X_1(\omega_i)$ at time 1, depending on the state ω_i of the market. The task is to calculate the time-0 price X_0 of the derivative.

Denote by ϕ^S and ϕ^B the number of units held in asset S and B, respectively. We call (ϕ^S, ϕ^B) a portfolio.

Definition 3.2.1. A derivative X is said to be *attainable* if there exists a portfolio (ϕ^S, ϕ^B) such that

$$\phi^S S_1(\omega_i) + \phi^B B_1(\omega_i) = X_1(\omega_i), \qquad \text{for each } i = 1, 2.$$

We call (ϕ^S, ϕ^B) a replicating portfolio for the derivative X. If all derivatives are attainable then the market is said to be complete. Otherwise, it is incomplete.

Next we introduce a key concept which is important both in determining whether or not the model is arbitrage-free and in valuing derivatives.

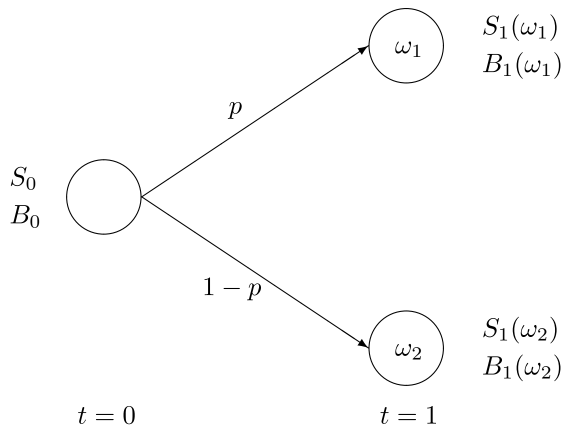


Figure 3.2: The possible states of the market.

Definition 3.2.2. A pricing kernel $\mathbf{Z} = (Z_1, Z_2)$ is any strictly positive vector with the property that

$$S_0 = \sum_{i=1}^{2} Z_i S_1(\omega_i) \quad \text{and} \quad B_0 = \sum_{i=1}^{2} Z_i B_1(\omega_i). \tag{3.2.1}$$

We see that the pricing kernel relates the time-1 prices to the time-0 prices. Each component Z_i corresponds to a state of the market ω_i.

Theorem 3.2.3. *The model is arbitrage-free if and only if there exists a pricing kernel.*

Theorem 3.2.4. *Suppose the model is arbitrage-free and let X be an attainable derivative. Then the time-0 value of X is given by*

$$X_0 = \phi^S S_0 + \phi^B B_0,$$

where (ϕ^S, ϕ^B) satisfies

$$\phi^S S_1(\omega_i) + \phi^B B_1(\omega_i) = X_1(\omega_i), \qquad \text{for each } i = 1, 2.$$

Furthermore, if $\mathbf{Z}$ is some pricing kernel for the model then X_0 can also be represented as

$$X_0 = \sum_{i=1}^{2} Z_i X_1(\omega_i). \tag{3.2.2}$$

Theorem 3.2.4 tells us that if we have a pricing kernel $\mathbf{Z}$ (which means by Theorem 3.2.3 that the model is arbitrage-free) then we can price any attainable derivative using the pricing kernel. We don't need to find the replicating portfolio for each derivative.

The usual interpretation of the pricing kernel is as a change of measure. To see this, without loss of generality suppose that the asset B is a bank account which pays interest at a continuously compounded rate of r. Further suppose that

$$B_0 = 1 \qquad \Rightarrow \qquad B_1(\omega_1) = B_1(\omega_2) = e^r.$$

We solve (3.2.1) to find

$$Z_1 = e^{-r} \frac{S_1(\omega_1) - e^r S_0}{S_1(\omega_1) - S_1(\omega_2)} \quad \text{and} \quad e^r Z_2 = 1 - e^r Z_1. \tag{3.2.3}$$

Defining

$$q := e^r Z_1,$$

we see from (3.2.3) and the strict positivity of the pricing kernel that $0 < q \leq 1$. We also find that $1 - q = e^r Z_2$. Substituting for q into (3.2.2), we get

$$X_0 = e^{-r} \sum_{i=1}^{2} e^r Z_i X_1(\omega_i) = e^{-r} \left(q X_1(\omega_1) + (1 - q) X_1(\omega_2) \right).$$

Interpreting q as a probability, we write this concisely as

$$X_0 = e^{-r} \mathrm{E}^{\mathbb{Q}} (X_1), \tag{3.2.4}$$

where we use $\mathrm{E}^{\mathbb{Q}}$ to denote expectation with respect to the measure $\mathbb{Q}$ which assigns probability q to state ω_1 and probability $1 - q$ to state ω_2. We call $\mathbb{Q}$ the *risk-neutral measure*. It does not depend on the risk preferences of the investor and it is in this context that the term "risk-neutral" should be understood; the measure $\mathbb{Q}$ is neutral with respect to risk preferences. We also call the formula (3.2.4) a *risk-neutral valuation* formula.

Equation (3.2.4) gives the fair value of the derivative as its time-1 expectation calculated using the risk-neutral measure $\mathbb{Q}$ and discounted to time-0. This means that under the measure $\mathbb{Q}$, the discounted asset price is a *martingale*.

Definition 3.2.5. In discrete-time, a *martingale* is a stochastic process $Y = \{Y_n; n = 0, 1, \ldots\}$ such that

$$\mathrm{E} \left(Y_n \,| \mathcal{F}_m \right) = Y_m, \quad \forall m \leq n,$$

where $\mathcal{F}_m$ denotes the information available at time m.

For this reason, the measure $\mathbb{Q}$ is also called a *martingale measure*. We can now state the no-arbitrage condition as follows.

Theorem 3.2.6. *The model is arbitrage-free if and only if there exists a martingale measure $\mathbb{Q}$.*

What is most suprising about the risk-neutral valuation formula is that it does not involve the measure $\mathbb{P}$ which assigns probability p to state ω_1 and probability $1 - p$ to state ω_2. We call $\mathbb{P}$ the *real-world measure*. It is under the measure $\mathbb{P}$ that we observe the asset prices; for example, with probability p we observe asset S's price at time 1 to be $S_1(\omega_1)$.

The only role of the measure $\mathbb{P}$ is to determine which events are possible and which are impossible. The martingale measure $\mathbb{Q}$ changes the probability of these events, but it must agree with the measure $\mathbb{P}$ on which events are impossible. If an event is impossible under measure $\mathbb{P}$ (that is, the event has probability zero) then it must also be impossible under measure $\mathbb{Q}$. More abstractly, the measure $\mathbb{P}$ determines a class of *equivalent probability measures*, which we see more of later.

Thus when we calculate the arbitrage-free price of a derivative, we do this in a risk-neutral world regardless of our actual risk preferences. The calculation holds for all investors. However, it must be remembered that the measure $\mathbb{Q}$ is an artificial construct which has no interpretation in the real-world. It is simply a means of obtaining the fair price of a derivative.

So far, we have not said anything about the uniqueness of the martingale measure.

Theorem 3.2.7. *Suppose the model is arbitrage-free. Then it is complete if and only if the martingale measure $\mathbb{Q}$ is unique.*

In summary, Theorems 3.2.6 and 3.2.7 tell us that the model is complete and arbitrage-free if and only if there exists a unique martingale measure $\mathbb{Q}$, that is a measure under which the discounted asset price is a martingale. Hence, while the measure $\mathbb{Q}$ has no interpretation in the real-world, its existence and uniqueness have very important implications for the model. In particular, if it is unique then (3.2.4) has a unique solution. If it is not unique, then there are multiple possible solutions to (3.2.4) and thus multiple possible prices for the derivative X.

Now we have reviewed the important concepts in a one period model, we turn to the continuous time model that is of most interest to us. We re-state the concepts in the continuous time model and the reader will see that the results carry over unchanged. The main difficulty in moving to continuous time is in the technical details. We omit many of these, but the interested reader can find them in many books, such as Björk (2009), Hunt and Kennedy (2004) and Karatzas and Shreve (1998).

3.3 Pricing in continuous time

We place the results from the previous section in a continuous time setting, for which it is essential to have a more probabilistic approach. Suppose we are given a market model with one traded asset (a stock) and one risk-free asset, and all processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the stock price process $S = \{S(t), t \in [0, T]\}$ follows the RSLN model, so that it satisfies

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(\alpha(t_-))\,\mathrm{d}t + \sigma(\alpha(t_-))\,\mathrm{d}W(t), \quad \forall t \in [0, T], \tag{3.3.1}$$

with the initial value $S(0)$ being a fixed, strictly positive constant, and we assume that the price dynamics of the risk-free asset satisfy

$$\frac{\mathrm{d}B(t)}{B(t)} = r(\alpha(t_-))\,\mathrm{d}t, \quad \forall t \in [0, T], \quad B(0) = 1.$$

The above price process dynamics are under the real-world measure $\mathbb{P}$.

The information available to the investors in the market at time t is the history of the Markov chain and Brownian motion up to and including time t.

Mathematically, this is represented by the filtration

$$\mathcal{F}_t := \sigma\{(\alpha(s), W(s)), s \in [0,t]\} \vee \mathcal{N}(\mathbb{P}), \quad \forall t \in [0,T], \tag{3.3.2}$$

where $\mathcal{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{F} = \mathcal{F}_T$.

Remark 3.3.1. The Markov chain and the Brownian motion are defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ and, as a mathematical consequence of this, they are independent processes (this is a result of Jacod (1979, Proposition 14.36, page 463)). Relating these processes to economic reality, we might think of the Brownian motion as modeling short-term, micro-economic changes in the market, whereas the Markov chain models long-term, macro-economic changes. With this interpretation, the implicit assumption in the RSLN model that these economic changes are independent is a reasonable approximation to reality. For practical implementation, this means that the number and specification of the market regimes should reflect this interpretation.

3.3.1 No-arbitrage and incompleteness

Definition 3.3.2. A *contingent claim with maturity date* T is a random variable $X \in \mathcal{F}_T$. A contingent claim X is called *simple* if it is of the form

$$X = \Phi(S(T), \alpha(T)),$$

for some given deterministic, measurable function Φ.

The technical requirement that $X \in \mathcal{F}_T$ means that at time T we have enough information to determine the amount of money that should be paid out. We only consider the valuation of simple contingent claims, which includes derivatives such as European options.

A portfolio is specified by the $\{\mathcal{F}_t\}$-predictable stochastic process $\phi = (\phi^S, \phi^B)$, where ϕ^S is the amount invested in the stock and ϕ^B is the amount invested in the bank account. The value of the portfolio at time t is denoted $V^\phi(t)$.

Definition 3.3.3. A portfolio ϕ with value $V^\phi(t)$ at time t is called *self-financing* if

$$\mathrm{d}V^\phi(t) = \phi^S(t)\,\mathrm{d}S(t) + \phi^B(t)\,\mathrm{d}B(t)$$

that is, a self-financing portfolio is a portfolio with no external infusions or withdrawals of money.

Definition 3.3.4. A contingent claim X with maturity date T is *attainable* if there exists a self-financing portfolio ϕ such that

$$V^\phi(T) = X, \quad \mathbb{P}\text{-a.s.}$$

We call ϕ a replicating portfolio for the contingent claim. The model is complete if all contingent claims are attainable and otherwise it is incomplete.

Before stating the no-arbitrage condition, we define precisely two terms which have already been introduced.

Definition 3.3.5. Given a measure $\mathbb{P}$, another probability measure $\mathbb{Q}$ on the same measurable space $(\Omega, \mathcal{F})$ is *equivalent* to $\mathbb{P}$ if

$$\mathbb{P}[A] > 0 \quad \iff \quad \mathbb{Q}[A] > 0, \qquad \forall A \in \mathcal{F}.$$

Definition 3.3.6. A stochastic process $Y = \{Y(t), t \in [0, T]\}$ is a *martingale* under the measure $\mathbb{Q}$ if $\mathrm{E}^{\mathbb{Q}}|Y(t)| < \infty$ for all $t \in [0, T]$ and

$$\mathrm{E}^{\mathbb{Q}}\left(Y(t) \,|\, \mathcal{F}_s\right) = Y(s), \quad \forall s \in [0, t].$$

Definition 3.3.7. A martingale measure is a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted stock price process $e^{-\int_0^t r(s)\,\mathrm{d}s} S(t)$ is a martingale under $\mathbb{Q}$.

Theorem 3.3.8. *The market model is free of arbitrage if and only if there exists a martingale measure.*

Theorem 3.3.9 (Risk-neutral valuation formula)**.** *Suppose the model is arbitrage-free. Then the fair price process for a contingent claim X with maturity date T is*

$$\Pi(t) = E^{\mathbb{Q}}\left(e^{-\int_t^T r(s)\,\mathrm{d}s} X \,|\, \mathcal{F}_t\right),$$

where $\mathbb{Q}$ is a martingale measure.

Remark 3.3.10. The risk-neutral valuation formula in continuous time is more complicated than in the one-period, discrete time model. However, it is interpreted the same way. To find the fair price for a contingent claim X, we discount the time-T payoff of the contingent claim back to time t. Then we condition on the information known at time t, as encapsulated by the notation $\mathcal{F}_t$, and find the expectation. Of course, this is done from the perspective that we are currently at time 0. We do not know exactly what information will be known at time t, so the filtration $\mathcal{F}_t$ is composed of all the events which could take place from time 0 to time t. For this reason, the price $\Pi(t)$ at time t is not a constant but a random variable, whose value depends on the events which could take place up to time t.

Theorem 3.3.11. *Suppose the model is arbitrage-free. Then it is complete if and only if the martingale measure $\mathbb{Q}$ is unique.*

The latter theorem exactly parallels Theorem 3.2.7, which applies in discrete time.

3.3.2 Incomplete market pricing

As we show shortly, the RSLN is incomplete. There have been various solutions proposed to cope with the non-uniqueness of derivative prices in incomplete markets. Here we review briefly some of the most well-known ones.

Obtaining a single price

As we have seen, in incomplete markets there does not generally exist a self-financing portfolio for every contingent claim X. Any portfolio which replicates the claim X will incur a cost, due to the hedging error arising from incompleteness. The portfolio which minimises the cost at every instant, among all possible replicating portfolios, is called a *locally risk-minimising portfolio*. Such portfolios can be characterized by a particular choice, called the *minimal martingale measure*, from the set of martingale measures. This idea was introduced by Follmer and Schweizer (1991). From a mathematical viewpoint, the minimal martingale measure is the measure which disturbs the structure of the probability space as little as possible, when moving from the real-world measure $\mathbb{P}$. From a financial viewpoint, it can be viewed as the measure which assigns a price of zero to the unhedgeable risk; in the RSLN model the unhedgeable risk is the risk of the Markov chain jumping between states.

Variations of the minimal martingale measure include the *minimal entropy martingale measure*, which is the martingale measure that minimises the relative entropy with respect to the real-world measure. Details can be found in Fritelli (2000).

While the minimal martingale measure is a popular pricing measure for financial mathematicians, the Esscher transform has been utilised as a means of pricing by some actuaries. The Esscher transform, a form of exponential tilting, of the logarithm of the stock price is used to select the *Esscher measure*. This option pricing method was introduced by Gerber and Shiu (1994) and it can be justified by the maximization of the power utility function of a representative agent.

There have been many other martingale measures proposed to select a single price in an incomplete market. We must remember, however, that it is the market who decides the price and not the individual investor. The market decides the martingale measure used for pricing, and it may be very different to the one we choose. However, if we can find a range of prices that the market price may reasonably be expected to lie in, we can analyse how much our chosen price can differ from the possible market prices. We can also use the range of prices as a guide to the reasonableness of our price from the market's perspective.

Narrowing the range of prices

To obtain a narrower range of pricing bounds than the no-arbitrage bounds, Bernardo and Ledoit (2000) considered the ratio of the expected value of positive payoffs to the expected value of negative payoffs, where the expectation is calculated with respect to a martingale measure, which they called the *gain-loss ratio*. They excluded measures which resulted in a gain-loss ratio which was too high.

The approach that we develop in the context of an RSLN model is the *good-deal bound approach*. Cochrane and Saá Requejo (2000) calculated the Sharpe Ratio of the assets in the market and excluded those martingale measures which implied a Sharpe Ratio which is too high. Essentially, they excluded those prices which imply an extreme compensation for the risks undertaken. The good-deal bound approach is attractive since historical data can be examined to determine which Sharpe Ratio is unreasonable (for example, Sharpe Ratios above 2 are unusual). This means that we can be more objective about the range of possible prices.

Ultimately, though, for contracts such as maturity guarantees we must choose a single price. The choice of the single price will depend on our attitudes to risk. However, as we mentioned above, the range of prices obtained through methods such as the good-deal bound approach can be used to guide our choice, for example by enabling us to avoid unreasonable prices. It also allows us to examine the sensitivity of the price to changes in the market attitude to risk, as expressed through the Sharpe Ratio in the case of the good-deal bound approach. It is for these reasons that we develop the good-deal bound approach in the RSLN market model.

3.4 Pricing in the RSLN model

Now that we have presented the fundamental theorems on asset pricing, we consider how they apply to the particular case of the RSLN model. First, the RSLN is free of arbitrage, which we can show concretely. Set

$$h(t) := -\frac{\mu(t) - r(t)}{\sigma(t)}$$

and define the *likelihood process* as the process L with dynamics

$$\begin{aligned} \mathrm{d}L(t) &= L(t)h(t)\,\mathrm{d}W(t) \\ L(0) &= 1. \end{aligned}$$

The solution is

$$L(t) = \exp\left\{ \int_0^t h(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t |h(s)|^2\,\mathrm{d}s \right\}.$$

Finally, define a new measure $\mathbb{Q}$ by the recipe

$$\mathbb{Q}[A] = \int_A L(T)\,\mathrm{d}\mathbb{P}, \qquad \forall A \in \mathcal{F}. \tag{3.4.1}$$

Now that we have concretely defined a measure $\mathbb{Q}$, we are in a position to determine the $\mathbb{Q}$-dynamics of the discounted stock price process. First consider the (non-discounted) stock price process, which has $\mathbb{P}$-dynamics

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(\alpha(t_-))\,\mathrm{d}t + \sigma(\alpha(t_-))\,\mathrm{d}W(t), \quad \forall t \in [0, T],$$

To switch to $\mathbb{Q}$-dynamics, we apply the *Girsanov theorem.* This allows is to switch between Brownian motion under $\mathbb{P}$-measure and $\mathbb{Q}$-measure, using the relation

$$\mathrm{d}W(t) = h(t)\,\mathrm{d}t + \mathrm{d}W^{\mathbb{Q}}(t),$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$-measure (recall that W is a standard Brownian motion under $\mathbb{P}$-measure). Substituting for $\mathrm{d}W(t)$ in the stock price dynamics results in

$$\begin{aligned} \frac{\mathrm{d}S(t)}{S(t)} &= \mu(t)\,\mathrm{d}t + \sigma(t)\left(h(t)\,\mathrm{d}t + \mathrm{d}W^{\mathbb{Q}}(t)\right) \\ &= \left(\mu(t) + \sigma(t)h(t)\right)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t) \\ &= r(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W^{\mathbb{Q}}(t). \end{aligned}$$

We apply the stochastic version of integration-by-parts to determine the dynamics of the discounted stock price process under $\mathbb{Q}$-measure.

$$\mathrm{d}\left(e^{-\int_0^t r(s)\,\mathrm{d}s} S(t)\right) = e^{-\int_0^t r(s)\,\mathrm{d}s} S(t)\,\sigma(t)\,\mathrm{d}W^{\mathbb{Q}}(t).$$

Thus the discounted stock price process is a martingale under $\mathbb{Q}$-measure. Since the method we used to define the measure $\mathbb{Q}$ in (3.4.1) necessarily means that $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we can apply Theorem 3.3.8 to conclude that

the RSLN model is arbitrage-free.

Next, we consider if the RSLN model is complete. Heuristically, the RSLN model has two sources of randomness - the Brownian motion and the Markov chain - but only one traded asset (the risk-free asset is not considered a traded asset). From this perspective, we do not expect the RSLN model to be complete. An example of a simple contingent claim which is not attainable is a contingent claim X with maturity date T which pays one unit if the Markov chain is in state 1, and otherwise zero units, that is

$$X = \begin{cases} 1 & \text{if } \alpha(T) = 1 \\ 0 & \text{if } \alpha(T) \neq 1. \end{cases} \tag{3.4.2}$$

Here, the payoff of X depends on the Markov chain and there is no replicating portfolio.

Theorem 3.3.11 tells us that if we can find more than one martingale measure then the RSLN model is incomplete. In Subsection 3.4.2 we construct the set $\mathcal{Q}$ of martingale measures for the RSLN model. However, to do that we require the martingales associated with the Markov chain; these are introduced in Subsection 3.4.1. For the moment, we simply state that

the RSLN model is incomplete,

since the martingale measure $\mathbb{Q}$ is not unique. This means that there is an interval

$$\left(\inf_{\mathbb{Q}\in\mathcal{Q}} \mathrm{E}^{\mathbb{Q}}\left(e^{-\int_t^T r(s)\,\mathrm{d}s} X \mid \mathcal{F}_t\right), \sup_{\mathbb{Q}\in\mathcal{Q}} \mathrm{E}^{\mathbb{Q}}\left(e^{-\int_t^T r(s)\,\mathrm{d}s} X \mid \mathcal{F}_t\right)\right)$$

of arbitrage-free prices for a contingent claim X. The end-points of this interval are called the *no-arbitrage bounds*. It is from the above range of prices that we choose either a single price - corresponding to one particular $\mathbb{Q} \in \mathcal{Q}$ - or a range of prices - corresponding to a subset $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ - for the contingent claim.

3.4.1 Markov chain martingales

To specify the set $\mathcal{Q}$ of martingale measures, we need a set of martingales corresponding to the Markov chain. We already have the martingales corresponding to the Brownian motion: the Brownian motion itself. Although the Markov chain α is not a martingale, we can find a set of canonical martingales which is associated with it. For each pair of distinct states (i, j) in the state space of the Markov chain, there is is a point process, or counting process,

$$N_{ij}(t) := \sum_{0<s\leq t} \mathbf{1}_{\{\alpha(s_-)=i\}}\,\mathbf{1}_{\{\alpha(s)=j\}}, \quad \forall t \in [0, T], \tag{3.4.3}$$

where $\mathbf{1}$ denotes the zero-one indicator function. The process $N_{ij}(t)$ counts the number of jumps that the Markov chain α has made from state i to state j up to time t. A realisation of $N_{ij}(t)$ is shown in Figures 3.3(b)-3.3(c), which corresponds to a realisation of the Markov chain shown in Figure 3.3(a).

Define the intensity process

$$\lambda_{ij}(t) := g_{ij}\,\mathbf{1}_{\{\alpha(t_-)=i\}}. \tag{3.4.4}$$

If we compensate $N_{ij}(t)$ by $\int_0^t \lambda_{ij}(s)\,\mathrm{d}s$, then the resulting process

$$M_{ij}(t) := N_{ij}(t) - \int_0^t \lambda_{ij}(s)\,\mathrm{d}s \tag{3.4.5}$$

is a martingale (see Rogers and Williams (2006, Lemma IV.21.12)). A realisation of the process $M_{ij}(t)$ is shown in Figures 3.3(d)-3.3(e). We refer to the set of martingales $\{M_{ij}; i,j \in \mathcal{I}, i \neq j\}$ as *the* $\mathbb{P}$*-martingales of* α.

3.4.2 Martingale measures

We determine the set $\mathcal{Q}$ of martingale measures for the RSLN model. To do this, fix $\mathbb{Q} \in \mathcal{Q}$ and define the likelihood process corresponding to the measure $\mathbb{Q}$ in the usual way as

$$L(t) := \mathrm{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\,\middle|\,\mathcal{F}_t\right), \quad \forall t \in [0,T].$$

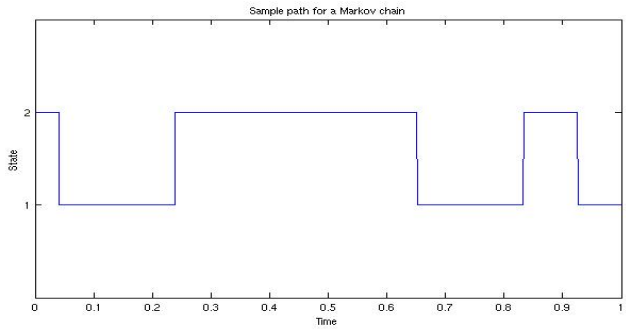
The standard notation $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. It can be thought of as a random variable which allows us to switch between $\mathbb{Q}$ and $\mathbb{P}$.

We can assume that $L(t)$ is a positive $\{\mathcal{F}_t\}$-martingale under the measure $\mathbb{P}$ (see Rogers and Williams (2006, Theorem IV.17.1)) with $L(0) = 1$, $\mathbb{P}$-a.s. Recalling that the filtration $\{\mathcal{F}_t\}$ is generated by both the Brownian motion W and the Markov chain α, we can apply an appropriate martingale representation theorem (for example, see Elliott (1976, Theorem 5.1)) to obtain predictable and suitably integrable stochastic processes $(h, \boldsymbol{\eta})$, for $\boldsymbol{\eta} := \{\eta_{ij};\, i,j = 1,\ldots,D,\, i \neq j\}$, satisfying

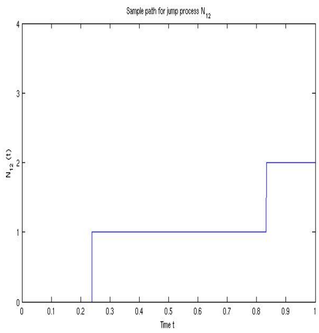
$$\frac{\mathrm{d}L(t)}{L(t_-)} = h(t)\,\mathrm{d}W(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \eta_{ij}(t)\,\mathrm{d}M_{ij}(t), \quad \forall t \in [0,T]. \tag{3.4.6}$$

Applying Protter (2005, Theorem 37, page 84) and using some algebra to find that the solution to the above equation is the stochastic exponential

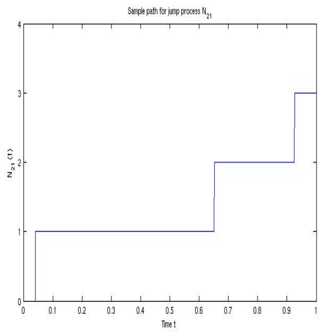
$$\begin{aligned} L(t) = \exp&\left\{\int_0^t h(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t |h(s)|^2\,\mathrm{d}s\right\} \\ &\cdot \exp\left\{\sum_{\substack{j=1,\\ j\neq i}}^{D}\int_0^t \eta_{ij}(s)\lambda_{ij}(s)\,\mathrm{d}s\right\} \prod_{s\in[0,t]}\prod_{j\neq i}\left(1 + \eta_{ij}(s)\Delta N_{ij}(s)\right), \end{aligned}$$



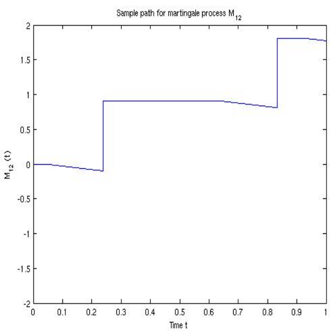
(a) A sample path of a 2-state Markov chain.



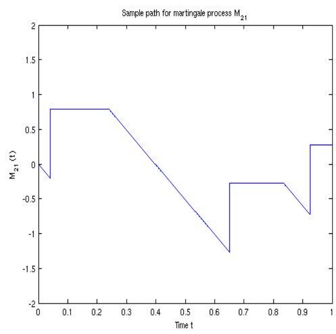
(b) Sample path of the jump process N_{12} associated with a 2-state Markov chain.



(c) Sample path of the jump process N_{21} associated with a 2-state Markov chain.



(d) Sample path of the martingale process M_{12} associated with a 2-state Markov chain.



(e) Sample path of the martingale process M_{21} associated with a 2-state Markov chain.

Figure 3.3: A sample path of a 2-state Markov chain and its associated jump processes and martingale processes.

where $\Delta N_{ij}(s) = 1$ if there is a jump in the Markov chain from state i to state j at time s, and otherwise $\Delta N_{ij}(s) = 0$. It is clear from the expression above that for $L(t)$ to be positive, the product must be positive which results in the requirement

$$\eta_{ij}(t) \geq -1, \qquad \forall j \neq i, \quad \forall t \in [0, T].$$

We call $(h, \boldsymbol{\eta})$ a *Girsanov kernel process.*

Remark 3.4.1. Given a Girsanov kernel $(h, \boldsymbol{\eta})$ process, we can generate the corresponding martingale measure $\mathbb{Q}$ as follows. Using (3.4.6) to define the likelihood process $L(t)$, we can recover the measure $\mathbb{Q}$ from (3.4.1). We say that the measure $\mathbb{Q}$ is *generated* by the Girsanov kernel process $(h, \boldsymbol{\eta})$. Thus the set $\mathcal{Q}$ of martingale measures for the RSLN model are those measures generated by the set of Girsanov kernel processes.

3.4.3 Changes of measure

The model tells us the price dynamics of the traded asset, the risk-free asset and the Markov chain under the measure $\mathbb{P}$. For the good-deal bound approach, we need to switch between the dynamics of the Brownian motion and the Markov chain's martingales under $\mathbb{P}$ and $\mathbb{Q}$. To do this, we use the Girsanov theorem. We have already come across the Girsanov theorem in Subsection 3.3.1 when we changed from a $\mathbb{P}$-Brownian motion to a $\mathbb{Q}$-Brownian motion. Now we need to apply the Girsanov theorem to the $\mathbb{P}$-martingales of the Markov chain. This gives us their dynamics under $\mathbb{Q}$ and relates it to their dynamics under $\mathbb{P}$. While the version of the Girsanov theorem which applies to Brownian motion is well-known, to apply it to the Markov chain's martingales we need the theorem's general form. This can be found, for example, in Protter (2005, Theorem 40, page 135).

Suppose we are given a Girsanov kernel process $(h, \boldsymbol{\eta})$. Generating a martingale measure $\mathbb{Q}$, we apply the Girsanov theorem to obtain the relationship

$$\mathrm{d}W(t) = h(t)\,\mathrm{d}t + \mathrm{d}W^{\mathbb{Q}}(t), \tag{3.4.7}$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. This means that W is no longer a Brownian motion when we consider it under the measure $\mathbb{Q}$; it is only a Brownian motion under the measure $\mathbb{P}$. For the $\mathbb{P}$-martingales of the Markov chain, the Girsanov theorem tells us that

$$\mathrm{d}M_{ij}(t) = \eta_{ij}(t)\lambda_{ij}(t)\,\mathrm{d}t + \mathrm{d}M_{ij}^{\mathbb{Q}}(t), \tag{3.4.8}$$

where the process $M_{ij}^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale, for each $j \neq i$. Substituting for M_{ij} from (3.4.5), we find

$$M_{ij}^{\mathbb{Q}}(t) = N_{ij}(t) - \int_0^t \left(1 + \eta_{ij}(s)\right)\lambda_{ij}(s)\,\mathrm{d}s, \quad \forall t \in [0, T]. \tag{3.4.9}$$

The set of martingales $\{M_{ij}^{\mathbb{Q}}; i, j \in \mathcal{I}, i \neq j\}$ are *the $\mathbb{Q}$-martingales of α.*

Remark 3.4.2. Compare the $\mathbb{Q}$-martingales of α to (3.4.5), which defines the $\mathbb{P}$-martingales of α. The point process N_{ij} is unaffected by the measure change. However, its compensator is $\int_0^t \left(1 + \eta_{ij}(s)\right)\lambda_{ij}(s)\,\mathrm{d}s$ under $\mathbb{Q}$, compared with being $\int_0^t \lambda_{ij}(s)\,\mathrm{d}s$ under $\mathbb{P}$. Recalling the definition of $\lambda_{ij}(t)$ from (3.4.4), this means that the generator of the Markov chain under the measure $\mathbb{Q}$ is the $D\times D$ matrix

$$G^{\mathbb{Q}}(t) = \left(g_{ij}\left(1+\eta_{ij}(t)\right)\right)_{i,j=1}^{D},$$

where we define the diagonal elements to be

$$\eta_{ii}(t) := -\sum_{j\neq i}\frac{g_{ij}}{g_{ii}}\left(1+\eta_{ij}(t)\right)-1.$$

As the Girsanov kernel process $(h,\boldsymbol{\eta})$ generating $\mathbb{Q}$ satisfies $\eta_{ij}(t)\geq -1$, for all $j\neq i$, and we already have the relations

$$g_{ij}\geq 0,\quad \forall j\neq i \qquad \text{and} \qquad g_{ii}=-\sum_{j\neq i}g_{ij},$$

we can conclude

$$g_{ij}\left(1+\eta_{ij}(t)\right)\geq 0,\quad \forall j\neq i \qquad \text{and} \qquad g_{ii}\left(1+\eta_{ii}(t)\right)=-\sum_{j\neq i}g_{ij}\left(1+\eta_{ij}(t)\right).$$

This means that, under $\mathbb{Q}$-measure, we cannot say that α is a Markov chain since its $\mathbb{Q}$-generator of α is a stochastic process. What we can say is that α is a *Markov process*, which means that it retains the Markov property.

3.4.4 Admissible Girsanov kernel processes

Here we develop restrictions on the Girsanov kernel processes that ensure the generated measure is a martingale measure.

Let $\mathbb{Q}$ be the measure generated by the Girsanov kernel process $(h,\boldsymbol{\eta})$. Consider an arbitrary asset in the market, with price process $\Pi=\{\Pi(t);t\in[0,T]\}$. Note that this asset is not restricted to the traded risky asset or risk-free asset, but it could be any derivative or self-financing strategy based on them and the Markov chain α. We know from Theorem 3.3.9 that the discounted price process of this arbitrary asset is an $\{\mathcal{F}_t\}$-martingale under the martingale measure $\mathbb{Q}$, that is

$$e^{-\int_0^t r(s)\,\mathrm{d}s}\Pi(t)=\mathrm{E}^{\mathbb{Q}}\left(e^{-\int_0^T r(s)\,\mathrm{d}s}\Pi(T)\,|\,\mathcal{F}_t\right).$$

As the filtration $\{\mathcal{F}_t\}$ is generated by both the Brownian motion and the Markov chain (recall (3.3.2)), we apply a suitable martingale representation theorem (such as Elliott (1976, Theorem 5.1)) to express this $\{\mathcal{F}_t\}$-martingale as the sum of a stochastic integral with respect to the $\mathbb{Q}$-Brownian motion and a stochastic integral with respect to the $\mathbb{Q}$-martingales of the Markov chain. To find the $\mathbb{P}$-dynamics, we use (3.4.7) and (3.4.8) to obtain the general $\mathbb{P}$-dynamics

$$\frac{\mathrm{d}\Pi(t)}{\Pi(t_-)}=\mu^{\Pi}(t)\,\mathrm{d}t+\sigma^{\Pi}(t)\,\mathrm{d}W(t)+\sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D}\gamma_{ij}^{\Pi}(t)\,\mathrm{d}M_{ij}(t). \tag{3.4.10}$$

The processes μ^{Π}, σ^{Π} and γ_{ij}^{Π} are suitably integrable and measurable with the condition, in order to avoid negative asset prices, that $\gamma_{ij}^{\Pi}(t)\geq -1$. At first, the form of the price process may seem surprising since they include a stochastic integral with respect to the $\mathbb{P}$-martingales of the Markov chain. However, if we consider an arbitrage asset which has terminal value which depends only on the Markov chain (for example, as in (3.4.2) then the inclusion of these martingales are natural. Note that if the asset is not the traded asset then the processes μ^{Π}, σ^{Π} and γ_{ij}^{Π} depend on the choice of the risk-neutral measure through the corresponding Girsanov kernel process.

Apply (3.4.7) and (3.4.8) to (3.4.10) to obtain the price dynamics Π of the arbitrarily chosen asset under the measure $\mathbb{Q}$:

$$\begin{aligned}\frac{\mathrm{d}\Pi(t)}{\Pi(t_-)} &= \left(\mu^{\Pi}(t) + \sigma^{\Pi}(t)h(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^{\Pi}(t)\eta_{ij}(t)\,\lambda_{ij}(t)\right)\mathrm{d}t \\ &\quad + \sigma^{\Pi}(t)\,\mathrm{d}W^{\mathbb{Q}}(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^{\Pi}(t)\,\mathrm{d}M_{ij}^{\mathbb{Q}}(t).\end{aligned} \tag{3.4.11}$$

The measure $\mathbb{Q}$ is a martingale measure if and only if the local rate of return of the asset under the measure $\mathbb{Q}$ equals the risk-free rate of return r. This follows from Theorem 3.3.8. Thus we obtain the following martingale condition, which is a condition on a potential Girsanov kernel process which ensures that it really does generate a martingale measure $\mathbb{Q}$.

Proposition 3.4.3. Martingale condition *The measure $\mathbb{Q}$ generated by the Girsanov kernel process $(h, \boldsymbol{\eta})$ is a martingale measure if and only if*

$$\eta_{ij}(t) \geq -1, \quad \forall j \neq i, \tag{3.4.12}$$

and for any asset in the market whose price process Π has $\mathbb{P}$-dynamics given by (3.4.10), we have

$$r(t) = \mu^{\Pi}(t) + \sigma^{\Pi}(t)h(t) + \sum_{i=1}^{D}\sum_{\substack{j=1,\\ j\neq i}}^{D} \gamma_{ij}^{\Pi}(t)\eta_{ij}(t)\,\lambda_{ij}(t), \quad \forall t \in [0,T]. \tag{3.4.13}$$

We refer to a Girsanov kernel process $(h, \boldsymbol{\eta})$ for which the generated measure $\mathbb{Q}$ is a martingale measure as an *admissible* Girsanov kernel process.

Remark 3.4.4. From (3.4.13) we have the following economic interpretation of an admissible Girsanov kernel process $(h, \boldsymbol{\eta})$: the process $-h$ is the market price of diffusion risk and $-\eta_{ij}$ is the market price of jump risk, or *regime change risk*, for a jump in the Markov chain from state i to state j.

Suppose we are given a Girsanov kernel process $(h, \boldsymbol{\eta})$ for which the generated measure $\mathbb{Q}$ is a martingale measure. The price dynamics under $\mathbb{P}$ of the underlying risky stock are as in (2.2.1), that is

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t), \quad \forall t \in [0,T].$$

By Proposition 3.4.3, we must have that

$$b(t) + \sigma(t)h(t) = r(t), \quad \forall t \in [0,T].$$

This means that the market price of diffusion risk $-h$ is determined by the price dynamics of the underlying risky asset. However, as there is no traded asset in the market which is based on the Markov chain α, we cannot say anything about the market price of jump risk $-\eta_{ij}$.

Chapter 4

Good-deal bounds

In this chapter, we apply the good-deal bound idea, first proposed by Cochrane and Saá Requejo (2000), to the RSLN model. The good-deal bound approach is a means of narrowing the no-arbitrage bounds, which can be too wide to be practically useful. The idea is to exclude those martingale measures which imply a Sharpe Ratio that is too high.

4.1 The Sharpe Ratio

4.1.1 The Sharpe Ratio of an arbitrary asset

We define a Sharpe Ratio process for an arbitrarily chosen asset, with $\mathbb{P}$-dynamics as in (3.4.10). Broadly, the Sharpe Ratio is the excess return above the risk-free rate of the asset per unit of risk. We make this definition precise in the RSLN model. As μ^{Π} is the local mean rate of return of the asset under the measure $\mathbb{P}$, we begin by defining the *risk premium process* R as

$$R(t) := \mu^{\Pi}(t) - r(t). \tag{4.1.1}$$

Next, we define a *volatility process* ν for the asset by

$$\mathrm{d}\langle \Pi, \Pi \rangle(t) = \Pi^2(t_-)\nu^2(t)\,\mathrm{d}t, \tag{4.1.2}$$

where $\langle \cdot, \cdot \rangle$ is the angle-bracket quadratic variation process, which can be interpreted as the variation of the process Π. Substituting for Π from (3.4.10), we obtain

$$\mathrm{d}\langle \Pi, \Pi \rangle(t) = \Pi^2(t_-) \left(|\sigma^{\Pi}(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1, \\ j \neq i}}^{D} |\gamma_{ij}^{\Pi}(t)|^2 \, \lambda_{ij}(t) \right) \mathrm{d}t \tag{4.1.3}$$

Comparing (4.1.2) and (4.1.3), we see that the squared volatility process satisfies

$$\nu^2(t) = |\sigma^{\Pi}(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1, \\ j \neq i}}^{D} |\gamma_{ij}^{\Pi}(t)|^2 \, \lambda_{ij}(t).$$

Recalling that the state space of the Markov chain α is denoted by $\mathcal{I} = \{1, \ldots, D\}$ and the intensity process $\lambda_{ij}(t)$ given by (3.4.4), define the norm $\|\cdot\|_{\lambda(t)}$ in the Hilbert space $L^2(\mathcal{I} \times \mathcal{I}, \lambda(t))$ by

$$\|\boldsymbol{\gamma}(t)\|^2_{\lambda(t)} := \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\gamma_{ij}(t)|^2 \, \lambda_{ij}(t).$$

Then we can write

$$\nu^2(t) = |\sigma^\Pi(t)|^2 + \|\boldsymbol{\gamma}^\Pi(t)\|^2_{\lambda(t)}.$$

Defining the Hilbert space

$$\mathcal{H} := \mathbb{R} \times L^2(\mathcal{I} \times \mathcal{I}, \lambda(t)), \tag{4.1.4}$$

and denoting by $\|\cdot\|_{\mathcal{H}}$ the norm in the Hilbert space $\mathcal{H}$, we can also express the volatility process as

$$\nu(t) = \|\big(\sigma^\Pi(t), \boldsymbol{\gamma}^\Pi(t)\big)\|_{\mathcal{H}}. \tag{4.1.5}$$

Finally, we are in a position to define the *Sharpe Ratio process* (SR) for the arbitrarily-chosen asset as

$$(SR)(t) := \frac{R(t)}{\nu(t)}. \tag{4.1.6}$$

The Sharpe Ratio process depends on the chosen asset's price process. We seek a bound that applies to *all* assets' Sharpe Ratio processes. To do this, we use the extended Hansen-Jagannathan inequality, which is derived in Björk and Slinko (2006) and is an extended version of the inequality introduced by Hansen and Jagannathan (1991).

4.1.2 An extended Hansen-Jagannathan Bound

Björk and Slinko (2006, Theorem A.1) extended the Hansen-Jagannathan Bound to a jump-diffusion market. We follow their proof to show that a similar result holds in the RSLN market model.

Lemma 4.1.1 (An extended Hansen-Jagannathan Bound)**.** *Recall the Hilbert space $\mathcal{H}$ in (4.1.4). For every admissible Girsanov kernel process $(h, \boldsymbol{\eta})$ and for any asset in the market whose price process Π has $\mathbb{P}$-dynamics given by (3.4.10) and, consequently, whose Sharpe Ratio process (SR) is given by (4.1.6), the following inequality holds.*

$$|(SR)(t)| \leq \|(h(t), \boldsymbol{\eta}(t))\|_{\mathcal{H}},$$

that is

$$|(SR)(t)|^2 \leq |h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\eta_{ij}(t)|^2 \, \lambda_{ij}(t), \tag{4.1.7}$$

where we recall the definition of the intensity process $\lambda_{ij}(t)$ from (3.4.4).

Proof. Fix an admissible Girsanov kernel process $(h, \boldsymbol{\eta})$ and an asset in the market whose price process Π has $\mathbb{P}$-dynamics given by (3.4.10). From (3.4.13), the martingale condition is

$$r(t) = \mu^{\Pi}(t) + \sigma^{\Pi}(t)h(t) + \sum_{i=1}^{D} \sum_{\substack{j=1, \\ j \neq i}}^{D} \gamma_{ij}^{\Pi}(t)\eta_{ij}(t)\,\lambda_{ij}(t), \quad \forall t \in [0, T].$$

Recalling the risk premium process is $R(t) = \mu^{\Pi}(t) - r(t)$, we can write the martingale condition as

$$-R(t) = \sigma^{\Pi}(t)h(t) + \sum_{i=1}^{D} \sum_{\substack{j=1, \\ j \neq i}}^{D} \gamma_{ij}^{\Pi}(t)\eta_{ij}(t)\,\lambda_{ij}(t), \quad \forall t \in [0, T].$$

As $\mathcal{H}$ is a Hilbert space, it has an inner product which we denote by $\langle \cdot\,, \cdot \rangle_{\mathcal{H}}$. We recognize the right-hand side of the equation above as an inner product, that is

$$-R(t) = \langle (\sigma^{\Pi}(t), \boldsymbol{\gamma}^{\Pi}(t)), (h(t), \boldsymbol{\eta}(t)) \rangle_{\mathcal{H}}.$$

From the Cauchy-Schwarz inequality, it is immediate that

$$|R(t)| \leq \|(\sigma^{\Pi}(t), \boldsymbol{\gamma}^{\Pi}(t))\|_{\mathcal{H}} \cdot \|(h(t), \boldsymbol{\eta}(t))\|_{\mathcal{H}}. \tag{4.1.8}$$

Next, from (4.1.6),

$$|(SR)(t)| = \frac{|R(t)|}{|\nu(t)|}.$$

Substituting from (4.1.5) and (4.1.8), we get the desired result:

$$|(SR)(t)| \leq \frac{\|(\sigma^{\Pi}(t), \boldsymbol{\gamma}^{\Pi}(t))\|_{\mathcal{H}} \cdot \|(h(t), \boldsymbol{\eta}(t))\|_{\mathcal{H}}}{\|(\sigma^{\Pi}(t), \boldsymbol{\gamma}^{\Pi}(t))\|_{\mathcal{H}}} = \|(h(t), \boldsymbol{\eta}(t))\|_{\mathcal{H}}.$$

□

From Lemma 4.1.1, we see that we can bound the Sharpe Ratios of all assets in the market by bounding the right-hand side of (4.1.7) by a constant.

4.2 The general problem

We consider the valuation of a simple contingent claim with maturity date T in the RSLN model. Denote the contingent claim by Z so that

$$Z := \Phi(S(T), \alpha(T)), \tag{4.2.1}$$

for a deterministic, measurable function Φ. As we have observed in Subsection 3.3.1, there is no unique martingale measure in the market and hence there is no unique price for the contingent claim. Rather than choosing one particular martingale measure to price the contingent claim, via some criteria, we seek instead to find a reasonable range of prices by excluding those martingale measures which imply Sharpe Ratios which are too high.

4.2.1 The good-deal bound

The key idea is that to restrict the set of martingale measures by way of the Sharpe Ratio, we use the Hansen-Jagannathan bound. Rather than bounding the Sharpe Ratios directly, we bound the right-hand side of (4.1.7) by a constant. We call the constant a *good-deal bound*.

Condition 4.2.1. There exists $B_0 \in \mathbb{R}$ such that

$$B_0 = \sup_{t\in[0,T]} |h(t)|^2, \qquad \text{a.s.}$$

Definition 4.2.2. A good-deal bound is a constant $B \geq B_0$.

Remark 4.2.3. Now we can see how a good-deal bound B is applied to bound the Sharpe Ratio process (SR) of any asset in the market:

$$|(SR)(t)|^2 \leq |h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\eta_{ij}(t)|^2 \, \lambda_{ij}(t) \leq B.$$

In other words, $|(SR)(t)| \leq \sqrt{B}$. The economic interpretation is that, under the good-deal bound approach, $\sqrt{B}$ is the highest achievable instantaneous Sharpe Ratio in the market and $-\sqrt{B}$ is the lowest achievable instantaneous Sharpe Ratio.

The good-deal bound, though, is actually a bound on the price $-\eta_{ij}$ of regime change risk, since the price $-h$ of diffusion risk is determined by the traded asset. Thus

$$\sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\eta_{ij}(t)|^2 \, \lambda_{ij}(t) \leq B - |h(t)|^2. \tag{4.2.2}$$

4.2.2 The good-deal bound price processes

We consider the problem of finding the upper and lower good-deal bounds on the range of possible prices of the contingent claim Z. We begin by finding the upper good-deal price process. To determine the price process, we utilise the risk-neutral pricing formula of Theorem 3.3.9. Instead of using the formula to calculate the price over all times $t \in [0,T]$ for a fixed martingale measure, we find at each time t the supremum over the martingale measures which satisfy the good-deal bound. Thus the upper good-deal price process does not correspond to one particular martingale measure, but depends on all of the possible martingale measures. The lower good-deal price process is similarly determined, except that we take the infimum rather than the supremum.

Definition 4.2.4. Suppose we are given a good-deal bound B. The *upper good-deal price process* V for the bound B is the optimal value process for the control problem

$$\sup_{(h,\boldsymbol{\eta})} \mathrm{E}^{\mathbb{Q}} \left(e^{-\int_t^T r(\tau)\,\mathrm{d}\tau} \Phi(S(T), \alpha(T)) \,\middle|\, \mathcal{F}_t \right), \tag{4.2.3}$$

where the predictable processes $(h, \boldsymbol{\eta})$ are subject to the constraints

$$h(t) = -(\sigma(t))^{-1}(b(t) - r(t)), \tag{4.2.4}$$

$$\eta_{ij}(t) \geq -1, \qquad \forall j \neq i, \tag{4.2.5}$$

and

$$|h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1,\\ j\neq i}}^{D} |\eta_{ij}(t)|^2 \lambda_{ij}(t) \leq B, \tag{4.2.6}$$

for all $t \in [0, T]$.

Definition 4.2.5. The *lower good-deal price process* V is defined as in Definition 4.2.4 except that "sup" in (4.2.3) is replaced by "inf".

Remark 4.2.6. The risk-neutral valuation formula in (4.2.3) implies that the local rate of return of the price process corresponding to the contingent claim $Z = \Phi(S(T), \alpha(T))$ equals the risk-free rate r under the measure $\mathbb{Q}$. The equality constraint (4.2.4) ensures that h is consistent with the market price of jump risk. Together with the constraint (4.2.5), these ensure that the measure $\mathbb{Q}$ generated by $(h, \boldsymbol{\eta})$ is a martingale measure, as in Proposition 3.4.3. Note that, due to the constant bound on $(h, \boldsymbol{\eta})$ in the constraint (4.2.6), the measure $\mathbb{Q}$ generated by $(h, \boldsymbol{\eta})$ is a martingale measure, and not just a local martingale measure.

4.3 Stochastic control approach

To find the upper and lower good-deal bounds, we use stochastic control techniques. This requires that the problem has a Markovian structure which is imposed through the following condition.

Condition 4.3.1. The supremum in (4.2.3) is taken over Girsanov kernel processes $(h, \boldsymbol{\eta})$ of the form

$$h(t) = h(t, S(t), \alpha(t_-)) \quad \text{and} \quad \eta_{ij}(t) = \eta_{ij}(t, S(t), \alpha(t_-)), \quad \forall j \neq i,$$

and $\eta_{ii}(t) = 0$, for all $t \in [0, T]$.

The condition ensures that the Markovian structure is preserved under a measure change to the martingale measure $\mathbb{Q}$ generated by the Girsanov kernel processes $(h, \boldsymbol{\eta})$.

Remark 4.3.2. We note from the constraint (4.2.4) that the process h is completely determined by the market parameters $r(t)$, $b(t)$ and $\sigma(t)$. This means that the requirement $h(t) = h(t, S(t), \alpha(t_-))$ is really a requirement that the market parameters are of the form

$$r(t) = r(t, S(t), \alpha(t_-)), \quad b(t) = b(t, S(t), \alpha(t_-)) \quad \text{and} \quad \sigma(t) = \sigma(t, S(t), \alpha(t_-)).$$

However, as we have already assumed that the market parameters are of a simpler form, the requirement that $h(t) = h(t, S(t), \alpha(t_-))$ is already satisfied.

4.3.1 The good-deal functions

Under Condition 4.3.1, the optimal expected value in (4.2.3) can be written as $V(t, S(t), \alpha(t_-))$ where the deterministic mapping $V : [0,T] \times \mathbb{R} \times \mathcal{I} \to \mathbb{R}$ is known as the *optimal value function*. From general dynamic programming theory (for example, see Björk (2009, Chapter 19)), the optimal value function satisfies the following Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t} + \sup_{(h,\boldsymbol{\eta})} \left\{ \mathbb{A}^{(h,\boldsymbol{\eta})} V \right\} - rV = 0 \tag{4.3.1}$$
$$V(T, x, i) = \Phi(x, i),$$

where the supremum in (4.3.1) is subject to the constraints (4.2.4) and (4.2.5). To expand the infinitesimal operator $\mathbb{A}^{(h,\boldsymbol{\eta})}$, we need a version of Itô's formula, which is given by Lemma 4.3.3.

Lemma 4.3.3. *Given a process x satisfying*

$$\begin{aligned} \mathrm{d}x_t &= b(t, x(t_-), \alpha(t_-))\,\mathrm{d}t + \sigma(t, x(t_-), \alpha(t_-))\,\mathrm{d}W(t) \\ x(t) &= x_0 \in \mathbb{R} \end{aligned} \tag{4.3.2}$$

and functions $V(\cdot,\cdot,i) \in C^2([0,T] \times \mathbb{R})$ for $i = 1, \ldots, D$, we have

$$\begin{aligned} V(t) = V(0) &+ \int_0^t \left(V_t(s_-) + V_x(s_-) b(s_-) + \frac{1}{2} V_{xx}(s_-)\sigma^2(s_-) \right) \mathrm{d}s \\ &+ \sum_{j=1}^{D} \int_0^t \left(V(s, x(s), j) - V(s_-, x(s_-), \alpha(s_-)) \right) g_{\alpha_{s_-}, j}\,\mathrm{d}s \\ &+ \int_0^t V_x(s_-)\sigma(s_-)\,\mathrm{d}W(s) \\ &+ \sum_{j \neq i} \int_0^t \left(V(s, x(s), j) - V(s_-, x(s_-), i) \right) \mathrm{d}M_{ij}(s). \end{aligned} \tag{4.3.3}$$

Proof. Denote the triple (t, x_t, α_t) by (t) and denote the triple $(t_-, x_{t_-}, \alpha_{t_-})$ by (t_-). Using x^c to denote the continuous part of a process x, apply Itô's formula (for example, see Protter (2005, Theorem V.18, page 278)) to the function V:

$$\begin{aligned} V(t) = V(0) &+ \int_0^t V_t(s_-)\,\mathrm{d}s + \int_0^t V_x(s_-)\,\mathrm{d}x(s) + \int_0^t V_\alpha(s_-)\,\mathrm{d}\alpha(s) \\ &+ \frac{1}{2}\int_0^t V_{xx}(s_-)\,\mathrm{d}[x^c, x^c](s) + \int_0^t V_{x\alpha}(s_-)\,\mathrm{d}[x^c, \alpha^c](s) \\ &+ \frac{1}{2}\int_0^t V_{xx}(s_-)\,\mathrm{d}[\alpha^c, \alpha^c](s) + \sum_{0<s\le t} \left(V(s) - V(s_-) - V_\alpha(s_-)\Delta\alpha(s) \right). \end{aligned} \tag{4.3.4}$$

Since α is constant between jumps, we have both $[\alpha^c, \alpha^c](t) = 0$ and $[x^c, \alpha^c](t) = 0$ for all $t \in [0,T]$. Moreover, since α is purely discontinuous, we have:

$$\int_0^t V_\alpha(s_-)\,\mathrm{d}\alpha(s) = \sum_{0<s\le t} V_\alpha(s_-)\Delta\alpha(s).$$

These two observations allow us to cancel out four terms on the right-hand side of (4.3.4), leaving

$$\begin{aligned} V(t) = V(0) &+ \int_0^t V_t(s_-)\,\mathrm{d}s + \int_0^t V_x(s_-)\,\mathrm{d}x(s) \\ &+ \frac{1}{2}\int_0^t V_{xx}(s_-)\,\mathrm{d}[x^c,x^c](s) + \sum_{0<s\le t}\left(V(s)-V(s_-)\right). \end{aligned} \tag{4.3.5}$$

Consider the last term in the above equation.

$$\begin{aligned} &\sum_{0<s\le t}\left(V(s)-V(s_-)\right) \\ =&\sum_{j\neq i}\sum_{0<s\le t}\left(V(s,x(s),j)-V(s_-,x(s_-),i)\right)\mathbf{1}_{\{\alpha_{s_-}=i\}}\mathbf{1}_{\{\alpha_s=j\}} \\ =&\sum_{j\neq i}\sum_{0<s\le t}\left(V(s,x(s),j)-V(s_-,x(s_-),i)\right)\Delta M_{ij}(s), \end{aligned}$$

where we used the fact that $\Delta M_{ij}(s) = \mathbf{1}_{\{\alpha_{s_-}=i\}}\mathbf{1}_{\{\alpha_s=j\}}$ in the last line. The last sum can then be written as an integral, replacing $\Delta M_{ij}(s)$ by $\mathrm{d}[M_{ij}](s)$, to give

$$\begin{aligned} &\sum_{0<s\le t}\left(V(s)-V(s_-)\right) \\ =&\sum_{j\neq i}\int_0^t \big(V(s,x_s,j)-V(s_-,x_{s_-},i)\big)\;\mathrm{d}[M_{ij}](s) \\ =&\sum_{j\neq i}\int_0^t \left(V(s,x(s),j)-V(s_-,x(s_-),i)\right)\;\mathrm{d}\left(M_{ij}(s)+\langle M_{ij}\rangle(s)\right) \\ =&\sum_{j\neq i}\int_0^t \left(V(s,x(s),j)-V(s_-,x(s_-),i)\right)\;\mathrm{d}M_{ij}(s) \\ &\qquad+\sum_{j=1}^{D}\int_0^t \left(V(s,x(s),j)-V(s_-,x(s_-),\alpha(s_-))\right)g_{\alpha_{s_-},j}\,\mathrm{d}s \end{aligned}$$

We plug this last expression for the sum $\sum_{0<s\le t}\left(V(s)-V(s_-)\right)$ and the dynamics of x from (4.3.2) into (2.2.1) to get

$$\begin{aligned} V(t) = V(0) &+ \int_0^t V_t(s_-)\,\mathrm{d}s + \int_0^t V_x(s_-)\left(b(s_-)\,\mathrm{d}s+\sigma(s_-)\,\mathrm{d}W(s)\right) \\ &+ \frac{1}{2}\int_0^t V_{xx}(s_-)\sigma^2(s_-)\,\mathrm{d}s + \sum_{j\neq i}\int_0^t \left(V(s,x(s),j)-V(s_-,x(s_-),i)\right)\;\mathrm{d}M_{ij}(s) \\ &+\sum_{j=1}^{D}\int_0^t \left(V(s,x(s),j)-V(s_-,x(s_-),\alpha(s_-))\right)g_{\alpha_{s_-},j}\,\mathrm{d}s. \end{aligned} \tag{4.3.6}$$

Rearranging, we obtain

$$\begin{aligned}V(t) &= V(0) + \int_0^t \left(V_t(s_-) + V_x(s_-)b(s_-) + \frac{1}{2}V_{xx}(s_-)\sigma^2(s_-) \right) \mathrm{d}s \\ &+ \sum_{j=1}^{D} \int_0^t \left(V(s, x(s), j) - V(s_-, x(s_-), \alpha(s_-)) \right) g_{\alpha_{s_-}, j} \, \mathrm{d}s \\ &+ \int_0^t V_x(s_-)\sigma(s_-) \, \mathrm{d}W(s) \\ &+ \sum_{j \neq i} \int_0^t \left(V(s, x(s), j) - V(s_-, x(s_-), \alpha(s_-)) \right) \mathrm{d}M_{ij}(s).\end{aligned} \tag{4.3.7}$$

□

The above lemma applied to the stock price process S, with $\mathbb{P}$-dynamics given by (2.2.1), gives the infinitesimal operator $\mathbb{A}^{(h,\boldsymbol{\eta})}$ as

$$\begin{aligned}&\mathbb{A}^{(h,\boldsymbol{\eta})}V(t,x,i) \\ &= r(t,x,i)x\frac{\partial V}{\partial x}(t,x,i) + \frac{1}{2}\sigma^2(t,x,i)x^2\frac{\partial^2 V}{\partial x^2}(t,x,i) \\ &+ \sum_{j=1}^{D} g_{ij}(1+\eta_{ij}(t)) \left(V(t,x,j) - V(t,x,i) \right),\end{aligned} \tag{4.3.8}$$

for all $(t,x,i) \in [0,T] \times \mathbb{R} \times \mathcal{I}$.

Having imposed Condition 4.3.1, we can re-cast the definitions of the good-deal bound price processes (Definitions 4.2.4 and 4.2.5) in the following form which is more amenable to solution.

Definition 4.3.4. Given a good-deal bound B, the *upper good-deal function* for the bound B is the solution to the following boundary value problem

$$\frac{\partial V}{\partial t}(t,x,i) + \sup_{(h,\boldsymbol{\eta})} \left\{ \mathbb{A}^{(h,\boldsymbol{\eta})}V(t,x,i) \right\} - r(t,i)V(t,x,i) = 0 \tag{4.3.9}$$

$$V(T,x,i) = \Phi(x,i),$$

where $\mathbb{A}^{(h,\boldsymbol{\eta})}$ is given by (4.3.8) and the supremum is taken over all functions $(h,\boldsymbol{\eta})$ subject to Condition 4.3.1 and satisfying

$$h(t,x,i) = -\left(\sigma(t,x,i)\right)^{-1}\left(b(t,x,i) - r(t,x,i)\right), \tag{4.3.10}$$

$$\eta_{ij}(t,x) \geq -1, \quad \forall j \neq i, \tag{4.3.11}$$

and

$$|h(t,x,i)|^2 + \sum_{\substack{j=1,\\ j\neq i}}^{D} g_{ij}|\eta_{ij}(t,x)|^2 \leq B, \tag{4.3.12}$$

for all $(t,x,i) \in [0,T] \times \mathbb{R} \times \mathcal{I}$. We denote the solution to (4.3.9) by V^{upper}.

Definition 4.3.5. The *lower good-deal function* is the solution to (4.3.9) but with the supremum replaced by an infimum, subject to Condition 4.3.1 and the constraints (4.3.10) - (4.3.12). We denote this solution by V^{lower}.

Thus, finding the upper good-deal bound reduces to solving the boundary value problem of (4.3.9) subject to the constraints. However, rather than attempting to solve (4.3.9) directly, we reduce it to two deterministic problems which we solve for each fixed triple $(t, x, i) \in [0, T] \times \mathbb{R} \times \mathcal{I}$.

Moreover, as h is completely determined by (4.3.10), we need to solve only for the optimal $\boldsymbol{\eta}$. Therefore, given h satisfying (4.3.10), we do the following.

1. Solve the static optimization problem of finding the optimal $\bar{\boldsymbol{\eta}}$ in

$$\sup_{(h,\boldsymbol{\eta})} \left\{ \mathbb{A}^{(h,\boldsymbol{\eta})} V(t, x, i) \right\},$$

subject to the constraints (4.3.11) and (4.3.12).

2. Using the optimal $\bar{\boldsymbol{\eta}}$ found above, solve the partial integro-differential equation ("PIDE")

$$\frac{\partial V}{\partial t} + \mathbb{A}^{(h,\bar{\boldsymbol{\eta}})} V - rV = 0 \tag{4.3.13}$$

$$V(T, x, i) = \Phi(x, i). \tag{4.3.14}$$

for V. The upper good-deal bound for the contingent claim Z is then given by the value $V(0, S(0), \alpha(0))$, where $S(0)$ is the initial stock price and $\alpha(0)$ is the initial state of the Markov chain (corresponding to the initial market regime).

The lower good-deal bound is found similarly, but with the supremum in the first step replaced by an infimum.

For the static optimization problem, we consider in more detail how to solve it in the next subsection. Having solved for the optimal $\bar{\boldsymbol{\eta}}$, we can then use numerical (computational) methods to solve the PIDE. A concrete example of this, where we calculate the good-deal bounds for a European put option (which is equivalent to a maturity guarantee), can be found in Chapter 5.

4.3.2 The static optimization problem

As we have seen above, the static optimization problem associated with the upper good-deal function of Definition 4.3.4 is to find for each triple $(t, x, i) \in [0, T] \times \mathbb{R} \times \mathcal{I}$ the optimal $\bar{\boldsymbol{\eta}}$ that attains the supremum of

$$\begin{aligned} &\mathbb{A}^{(h,\boldsymbol{\eta})} V(t, x, i) \\ &= r(t, x, i) x \frac{\partial V}{\partial x}(t, x, i) + \frac{1}{2}\sigma^2(t, x, i) x^2 \frac{\partial^2 V}{\partial x^2}(t, x, i) \\ &+ \sum_{j=1}^{D} g_{ij} (1 + \eta_{ij}(t, x)) \left(V(t, x, j) - V(t, x, i) \right), \end{aligned} \tag{4.3.15}$$

subject to the constraints

$$\eta_{ij}(t) \geq -1, \quad \forall j \neq i \quad \text{and} \quad \sum_{\substack{j=1, \\ j \neq i}}^{D} g_{ij} |\eta_{ij}(t)|^2 \leq B - |h(t, x, i)|^2, \tag{4.3.16}$$

for h given by (4.3.10).

As the only term in (4.3.15) which involves $\boldsymbol{\eta}$ is the last one, we can equivalently consider the problem of finding the optimal $\bar{\boldsymbol{\eta}}$ which attains the supremum of

$$\sum_{j=1}^{D} g_{ij}(1+\eta_{ij}(t,x))\left(V(t,x,j)-V(t,x,i)\right), \tag{4.3.17}$$

subject to the constraints in (4.3.16). This is a linear optimization problem with both linear and quadratic constraints. Such problems are standard in optimization theory and, if it is too time-consuming to find the analytic solution, an algorithm can be used to calculate the solution numerically.

However, we are typically only interested in the cases when the number of regimes is small, i.e. only two or three regimes. In particular, when there are only two states of the Markov chain, we can find an analytic solution by considering the sign of $V(t,x,j)-V(t,x,i)$ in (4.3.17), subject to the constraints in (4.3.16).

Lemma 4.3.6. *For the RSLN(2) model, define for $i=1,2$,*

$$\tilde{B}(t,x,i) := \sqrt{\frac{B-|h(t,x,i)|^2}{-g_{ii}}}.$$

Then the solution to the static optimization problem associated with the upper good-deal function of Definition 4.3.4 is

$$\bar{\eta}_{ij}(t,x) = \begin{cases} \tilde{B}(t,x,i) & \text{if } V(t,x,j)-V(t,x,i)>0 \\ -\min\left[1,\tilde{B}(t,x,i)\right] & \text{if } V(t,x,j)-V(t,x,i)\le 0, \end{cases}$$

and the solution to the static optimization problem associated with the lower good-deal function of Definition 4.3.5 is

$$\bar{\eta}_{ij}(t,x) = \begin{cases} -\min\left[1,\tilde{B}(t,x,i)\right] & \text{if } V(t,x,j)-V(t,x,i)>0 \\ \tilde{B}(t,x,i) & \text{if } V(t,x,j)-V(t,x,i)\le 0. \end{cases}$$

Proof. Suppose $i=1$. The static optimization problem for the fixed triple $(t,x,1)$ is to find the supremum of

$$g_{12}(1+\eta_{12}(t,x))\left(V(t,x,2)-V(t,x,1)\right), \tag{4.3.18}$$

subject to the constraints

$$\eta_{12}(t)\ge -1, \quad \text{and} \quad g_{12}|\eta_{12}(t)|^2 \le B-|h(t,x,1)|^2.$$

As $g_{12}=-g_{11}$, the second constraint gives the inequality

$$-\tilde{B}(t,x,1)\le \eta_{12}(t)\le \tilde{B}(t,x,1).$$

Combining this with the first constraint $\eta_{12}(t)\ge -1$, we get

$$-\min\left[1,\tilde{B}(t,x,1)\right]\le \eta_{12}(t)\le \tilde{B}(t,x,1). \tag{4.3.19}$$

Now consider maximising (4.3.18). As $g_{12} > 0$ then we need only consider the sign of $V(t,x,2) - V(t,x,1)$. If $V(t,x,2) - V(t,x,1) > 0$ then we maximise $1+\eta_{12}(t,x)$ subject to the constraint (4.3.19). This immediately leads to $\eta_{12}(t) = \tilde{B}(t,x,1)$. If $V(t,x,2) - V(t,x,1) \leq 0$ then we minimise $1+\eta_{12}(t,x)$ subject to the constraint (4.3.19). This gives $\eta_{12}(t) = -\min[1, \tilde{B}(t,x,1)]$.

For the lower good-deal function, we consider minimising (4.3.18). If $V(t,x,2) - V(t,x,1) > 0$ then we minimise $1+\eta_{12}(t,x)$ subject to the constraint (4.3.19). This gives $\eta_{12}(t) = -\min[1, \tilde{B}(t,x,1)]$. If $V(t,x,2)-V(t,x,1) \leq 0$ then we maximise $1+\eta_{12}(t,x)$ subject to the constraint (4.3.19), whence $\eta_{12}(t) = \tilde{B}(t,x,1)$.

By symmetry, we find the corresponding results for the fixed triple $(t,x,2)$. Hence we have solved the static optimization problem for both the upper and lower good-deal functions when there are only two market regimes.

□

4.4 Minimal martingale measure

Here we leave aside the good-deal bounds and consider the minimal martingale measure, which was introduced by Follmer and Schweizer (1991). Recall from Remark 3.4.4 that $-\eta_{ij}$ is the market price of regime change risk, for a change in regime from i to j corresponding to a jump in the Markov chain from state i to state j. In the RSLN model, the market price of regime change risk is not determined by the traded asset. It is the martingale measure used for pricing which decides the market price of regime change risk and the minimal martingale measure assigns it value zero. For this reason, we consider the minimal martingale measure as a benchmark measure for pricing any derivative in the RSLN market.

Definition 4.4.1. The *minimal martingale measure* is the measure $\mathbb{Q}^{\min}$ generated by $(h^{\min}, \boldsymbol{\eta}^{\min})$, where $(h^{\min}, \boldsymbol{\eta}^{\min})$ is the Girsanov kernel process which minimizes

$$|h(t)|^2 + \sum_{i=1}^{D} \sum_{\substack{j=1, \\ j \neq i}}^{D} |\eta_{ij}(t)|^2 \, \lambda_{ij}(t)$$

subject to the constraint

$$b(t) + \sigma(t)h(t) = r(t), \quad \forall t \in [0,T].$$

It is immediate that the minimal martingale measure $\mathbb{Q}^{\min}$ is generated by

$$h^{\min}(t) := -\sigma^{-1}(t)\,(b(t) - r(t)) \qquad \text{and} \qquad \eta_{ij}^{\min}(t) := 0, \quad \forall j \neq i,$$

for all $t \in [0,T]$. As $\eta_{ij}^{\min}(t) \geq -1$, we have that $(h^{\min}, \boldsymbol{\eta}^{\min})$ is an admissible Girsanov kernel process.

Remark 4.4.2. Under the measure $\mathbb{Q}^{\min}$, the process α is a Markov chain with the same generator matrix $G = (g_{ij})_{i,j=1}^{D}$ as under the measure $\mathbb{P}$. In particular, this means that the measure $\mathbb{Q}^{\min}$ preserves the martingale property of the process $M_{ij}(t)$ defined by (3.4.5), so that the $\mathbb{P}$-martingales of α are also its $\mathbb{Q}^{\min}$-martingales.

Notice that $(h^{\min}, \boldsymbol{\eta}^{\min})$ minimizes the right-hand side of (4.1.7) over the set of admissible Girsanov kernel processes. Moreover, by Definition 4.2.2, any good-deal bound B satisfies $B \geq B_0$. This means

$$B \geq B_0 = \sup_{t \in [0,T]} |h(t)|^2 = \sup_{t \in [0,T]} |\big(h^{\min}(t), \boldsymbol{\eta}^{\min}(t)\big)|^2.$$

Thus $(h^{\min}, \boldsymbol{\eta}^{\min})$ is a Girsanov kernel process which satisfies the good-deal bound constraint in (4.3.12).

Denote the solution to the PIDE

$$\frac{\partial V}{\partial t} + \mathbb{A}^{(h^{\min}, \boldsymbol{\eta}^{\min})} V - rV = 0 \tag{4.4.1}$$

$$V(T, x, i) = \Phi(x, i) \tag{4.4.2}$$

by $V^{\min}$. Then as $(h^{\min}, \boldsymbol{\eta}^{\min})$ is a Girsanov kernel process which satisfies the good-deal bound constraint in (4.3.12), it is clear from this and Definitions 4.3.4 and 4.3.5 that the following relation holds:

$$V^{\text{lower}} \leq V^{\min} \leq V^{\text{upper}}.$$

When we determine the good-deal bounds for the example in Chapter 5, we see the empirical evidence of the latter inequality.

Chapter 5

Numerical examples

In Chapter 4, we developed the good-deal bound approach in the context of a RSLN model. The next question which naturally arises is: are they of any use? Perhaps they are still too wide to be practically useful. In this chapter, we calculate the good-deal bounds for a maturity guarantee. We ignore mortality but this could be easily included by multiplying the upper and lower good-deal bounds by the probability of survival from the policy issue date until the maturity date.

5.1 Setup

RSLN market model

For the numerical examples, we use an RSLN(2) market model, so that there are two market regimes. We assume that time is measured in years. In Hardy (2003, page 226), a regime-switching model was fitted to data from the FTSE All-Share Total Return Index from 1956 to 2001. Based on these figures, we use the market parameters for the risky asset in Table 5.1 and we take the generator of the Markov chain to be

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} -0.15 & 0.15 \\ 2 & -2 \end{pmatrix}.$$

From the generator, we see that the average time spent in regime 1 is about 6.7 years and the average time spent in regime 2 is 6 months. For the risk-free rate of return r, we use the average of the Bank of England bank base rate over the time 1956 to 2001, and we do not distinguish between market regimes. The reason is that the risk-free interest rate should not be strongly dependent on

Table 5.1: Market parameters

Regime i	$r(i)$	$\mu(i)$	$\sigma(i)$
1	0.085	0.155	0.15
2	0.085	-0.155	0.46

the stock market regimes (though we may expect some dependence, we ignore this for simplicity).

For this market model, the market price of diffusion risk is

$$h(1) = 0.467 \quad \text{and} \quad h(2) = -0.522,$$

varying with the market regime.

Maturity guarantee

We consider the problem of finding the upper and lower good-deal bounds for a maturity guarantee. As we saw in Chapter 1, guaranteeing the maturity value of a contract by an amount K is equivalent to the insurer selling an embedded European put option with strike price K. At the maturity date, the payout from the insurer is

$$\max\left[K - S(T), 0\right].$$

Therefore, we seek to calculate the upper and lower good-deal bounds for a T-year European put option with strike price K for a good-deal bound B. We consider values of $T \in \{3, 5, 10\}$ and a fixed strike price $K = 100$. Any good-deal bound B must satisfy the constraint (recall Condition 4.2.1 and Definition 4.2.2)

$$B \geq B_0 = \max[h^2(1), h^2(2)] = \max[(0.467)^2, (-0.522)^2] = 0.272.$$

Calculating the good-deal price bounds

Recall from Section 4.3 that to find the upper and lower good-deal bounds 4.3.5 we

- solve the associated static optimization problem, and then
- numerically solve the PIDE (4.3.13)-(4.3.14) using the solution to the static optimization problem.

We have already solved the static optimization problem for an RSLN(2) model, with the solution given by Lemma 4.3.6. Then it remains to numerically solve the PIDE

$$\begin{aligned}\frac{\partial V}{\partial t}(t,x,i) + r(i)x\frac{\partial V}{\partial x}(t,x,i) + \frac{1}{2}\sigma^2(i)x^2\frac{\partial^2 V}{\partial x^2}(t,x,i) - r(i)V(t,x,i)&\\ -g_{ii}(1+\bar{\eta}_{ij}(t,x))\left(V(t,x,j) - V(t,x,i)\right) = 0&\qquad(5.1.1)\\ V(T,x,i) = \max[K-x, 0],&\end{aligned}$$

using the optimal values $\bar{\eta}_{ij}$ given by Lemma 4.3.6. We outline the fully implicit finite difference method that we used to solve the PIDE in the Appendix.

Denoting the solution to the PIDE by V^{upper} for the upper good-deal bound and by V^{lower} for the lower good-deal bound, the good-deal price range for the T-year European put option with strike price K is $(V^{\text{lower}}(0), V^{\text{upper}}(0))$.

5.2 Results

We consider three European put options, each with strike price $K = 100$ and with maturities of 3, 5 and 10 years.

Fixed choice of the good-deal bound B

We began the study by choosing the good-deal bound $B = 0.3$. The choice corresponds to considering only those martingale measures which imply a Sharpe Ratio in the range $[-\sqrt{0.3}, \sqrt{0.3}] \approx [-0.55, 0.55]$ for all the assets in the economy. It also corresponds to the market price of regime change risk satisfying the bounds

$$-0.739 \leq \eta_{12}(t, x) \leq 0.739 \quad \text{and} \quad -0.117 \leq \eta_{21}(t, x) \leq 0.117,$$

where the numbers have been obtained by consideration of (4.2.2). We calculate the upper and lower good-deal price bounds for the European put options for various initial stock prices, as well as the minimal martingale measure price (recall that the minimal martingale measure assigns price zero to the market price of regime change risk and is the benchmark price).

The results for the 3-year put option are shown in Table 5.2. The first two columns give the initial conditions, consisting of the stock price and the market regime at time 0. In the next three columns are the good-deal price bounds and the minimal martingale measure ("MMM") price. The last three columns give the width of the good-deal price bounds and their ratio to the minimal martingale measure price.

We begin by making some general observations, which are not specific to the good-deal bound approach. The first thing to notice is the impact of the initial market regime on the prices. Comparing the prices for a fixed initial stock price $S(0)$, they are lower when the market starts in regime 1 (the *regime-1 price*) than when the market starts in regime 2 (the *regime-2 price*). This can be explained by the average time that the market spends in each regime and the market parameters of each regime. The average time spent in regime 1 is 6.7 years, so if the market starts in regime 1 then, after 3 years, it is likely to still be in regime 1. Under regime 1, the terminal stock price $S(T)$ is likely to have increased, since the mean rate of return of the traded asset is positive (recall the market parameters in Table 5.1). This means that there is less chance of a payout being made for the put option. Suppose now that the market starts in regime 2. As it spends on average 6 months in regime 2, then we expect that after 3 years the market has exited the starting regime 2 and is now in regime 1, where it spends around 6.7 years. As the mean rate of return of the traded asset is negative in regime 2, we expect that the terminal stock price $S(T)$ is lower than if the market had started in regime 1. The end result is that there is a higher chance of a payout for the option and, hence, a higher put option price.

Next we notice that for a fixed initial market regime, the put option prices decrease as the initial stock price increases. When the initial stock price is below the strike price (that is, it is *in-the-money*), the chance of the option being exercised, and hence having a positive payoff, is increased. This is reflected in the option prices. When the initial stock price is above the strike price (that is, it is *out-of-the-money*), there is less chance of the option being exercised and so

Table 5.2: Three year European put option, $K = 100$, $B = 0.3$

Initial stock price	Initial market regime	Lower good-deal price	MMM price	Upper good-deal price	Good-deal bound width	Lower / MMM	Upper / MMM
75	1	9.5559	10.5803	11.6317	2.0758	0.903	1.099
80	1	7.3208	8.3786	9.4639	2.1431	0.874	1.130
85	1	5.5361	6.5909	7.6762	2.1401	0.840	1.165
90	1	4.1420	5.1639	6.2211	2.0791	0.802	1.205
95	1	3.0735	4.0403	5.0481	1.9746	0.761	1.249
100	1	2.2672	3.1644	4.1082	1.8410	0.716	1.298
105	1	1.6667	2.4862	3.3573	1.6906	0.670	1.350
110	1	1.2239	1.9627	2.7572	1.5333	0.624	1.405
115	1	0.8997	1.5587	2.2762	1.3766	0.577	1.460
120	1	0.6636	1.2462	1.8887	1.2251	0.532	1.516
125	1	0.4919	1.0030	1.5743	1.0824	0.490	1.570
75	2	12.9527	13.9403	14.9639	2.0112	0.929	1.073
80	2	10.8309	11.8497	12.9051	2.0742	0.914	1.089
85	2	9.0377	10.0632	11.1272	2.0895	0.898	1.106
90	2	7.5352	8.5466	9.5993	2.0641	0.882	1.123
95	2	6.2842	7.1484	8.2902	2.0060	0.879	1.160
100	2	5.2472	6.1847	7.1702	1.9230	0.848	1.159
105	2	4.3897	5.2753	6.2121	1.8224	0.832	1.178
110	2	3.6811	4.5094	5.3914	1.7103	0.816	1.196
115	2	3.0952	3.8633	4.6868	1.5916	0.801	1.213
120	2	2.6098	3.3167	4.0800	1.4702	0.787	1.230
125	2	2.2063	2.8525	3.5554	1.3491	0.773	1.246

there is less chance of a positive payoff. Thus the option prices decrease as the put option moves from being in-the-money to *at-the-money* (when the initial stock price equals the strike price) to out-of-the-money.

As expected, the minimal martingale measure price always lies between the lower and upper good-deal price bounds. It is approximately halfway between the lower and upper good-deal price bounds; the minimal martingale measure assigns zero price to the regime change risk $\eta_{ij}(t)$ whereas in the calculation of the good-deal bounds, the martingale measures considered can assign positive or negative prices, as long as the good-deal bound is obeyed (recall (4.2.2)).

Now we examine the good-deal price bounds, beginning with the regime-1 prices. When the initial stock price is 75, so that the option is deeply in-the-money, the good-deal price bounds are about 10% below and above the minimal martingale measure price of 10.5803. As the initial stock price rises, these ratios increase to approximately 30% of the minimal martingale measure price of 4.1082 when the option is at-the-money. The ratios continue to increase as the option moves to being out-of-the-money, reaching a peak of 51% above and 57% below the minimal martingale measure price of 1.5743 when the initial stock price is 125. In comparison, from the third-last column, we see in that the absolute width of the good-deal price bound roughly decreases. This can be interpreted as follows. When the option is deeply in-the-money, so that a payout is more likely, the market price of regime change risk has a greater effect on the option price. This is reflected in a large absolute width of the good-deal price bounds. However, due to the high prices of the option, the width relative to the minimal martingale price is smaller. Conversely, when the option is deeply out-of-the-money, so that a payout is much less likely, the impact of the market price of regime change risk on the option prices is diminished. This is reflected in a smaller absolute width of the good-deal price bounds but, as the option prices are correspondingly small, the relative width is larger.

For the regime-2 prices, we see a similar pattern emerging. The absolute width of the regime-2 good-deal price bounds are slightly higher than the corresponding regime-1 good-deal price bounds. However, due to the higher option prices, the ratios of the good-deal price bounds to the minimal martingale measure prices are lower.

We also calculated the good-deal price bounds for a five year European put option and a ten year European put option; these results are shown in Table 5.3 and Table 5.4. The comments made for the three year option apply to both these longer-dated contracts. By comparing the values in all three tables for a fixed initial market regime and initial stock price, we notice that as the maturity increases, the option prices decrease. As the time to maturity increases, the market is more likely to have spent time in regime 1, in which the traded asset has a positive mean rate of return. Thus the probability of a payout decreases, as do the prices. However, the variation of the width of the good-deal pricing bounds is not as clear-cut: it is widest for the five year option, then mostly it is next widest for the three year option. This is probably a result of the lower prices at longer maturities being offset by the price of the regime change risk imposed by the good-deal bound.

For a fixed initial stock price, as the maturity increases the regime-1 and regime-2 prices start to converge. This reflects the Markov chain which models the regime-switching tending towards its stationary distribution. We would expect that for much higher maturities, for example 15 year contracts, the

Table 5.3: Five year European put option, $K = 100$, $B = 0.3$

Initial stock price	Initial market regime	Lower good-deal price	MMM price	Upper good-deal price	Good-deal bound width	Lower / MMM	Upper / MMM
75	1	5.7616	7.0608	8.3833	2.6217	0.816	1.187
80	1	4.4912	5.7514	7.0430	2.5518	0.781	1.225
85	1	3.4899	4.6889	5.9284	2.4385	0.744	1.264
90	1	2.7073	3.8299	5.0026	2.2953	0.707	1.306
95	1	2.0992	3.1366	4.2331	2.1339	0.669	1.350
100	1	1.6286	2.5771	3.5920	1.9634	0.632	1.394
105	1	1.2655	2.1247	3.0563	1.7908	0.596	1.438
110	1	0.9856	1.7578	2.6066	1.6210	0.561	1.483
115	1	0.7698	1.4590	2.2272	1.4574	0.528	1.527
120	1	0.6031	1.2144	1.9053	1.3022	0.497	1.569
125	1	0.4741	1.0130	1.6306	1.1565	0.468	1.610
75	2	8.2432	9.4778	10.7430	2.4998	0.870	1.133
80	2	6.8956	8.1109	9.3616	2.4660	0.850	1.154
85	2	5.7738	6.9520	8.1714	2.3976	0.831	1.175
90	2	4.8422	5.9702	7.1455	2.3033	0.811	1.197
95	2	4.0693	5.1380	6.2598	2.1905	0.792	1.218
100	2	3.4281	4.4318	5.4937	2.0656	0.774	1.240
105	2	2.8955	3.8310	4.8291	1.9336	0.756	1.261
110	2	2.4522	3.3186	4.2507	1.7985	0.739	1.281
115	2	2.0823	2.8710	3.7454	1.6631	0.725	1.305
120	2	1.7725	2.5030	3.3022	1.5297	0.708	1.319
125	2	1.5120	2.1776	2.9118	1.3998	0.694	1.337

Table 5.4: Ten year European put option, $K = 100$, $B = 0.3$

Initial stock price	Initial market regime	Lower good-deal price	MMM price	Upper good-deal price	Good-deal bound width	Lower / MMM	Upper / MMM
75	1	1.8759	2.9151	4.0134	2.1375	0.644	1.377
80	1	1.5002	2.4515	3.4738	1.9736	0.612	1.417
85	1	1.2044	2.0691	3.0144	1.8100	0.582	1.457
90	1	0.9707	1.7521	2.6213	1.6506	0.554	1.496
95	1	0.7852	1.4878	2.2832	1.4980	0.528	1.535
100	1	0.6375	1.2664	1.9909	1.3534	0.504	1.572
105	1	0.5192	1.0799	1.7370	1.2178	0.481	1.608
110	1	0.4241	0.9220	1.5155	1.0914	0.460	1.644
115	1	0.3472	0.7876	1.3214	0.9742	0.441	1.678
120	1	0.2848	0.6727	1.1506	0.8658	0.423	1.710
125	1	0.2338	0.5740	0.9998	0.7660	0.407	1.742
75	2	2.8312	3.8939	5.0062	2.1750	0.727	1.286
80	2	2.3728	3.3655	4.4173	2.0445	0.705	1.313
85	2	1.9967	2.9185	3.9073	1.9106	0.684	1.339
90	2	1.6870	2.5386	3.4637	1.7767	0.665	1.364
95	2	1.4307	2.2143	3.0761	1.6454	0.646	1.389
100	2	1.2177	1.9360	2.7358	1.5181	0.629	1.590
105	2	1.0398	1.6962	2.4357	1.3959	0.613	1.436
110	2	0.8906	1.4886	2.1701	1.2795	0.598	1.458
115	2	0.7648	1.3079	1.9338	1.1690	0.585	1.479
120	2	0.6582	1.1500	1.7228	1.0646	0.572	1.498
125	2	0.5674	1.0113	1.5336	0.9662	0.561	1.516

regime-1 and regime-2 prices are almost identical since the initial market regime becomes much less important.

Figures 5.1 - 5.3 show graphically the upper and lower good-deal bounds for the fixed choice of good-deal bound $B = 0.3$, as well as the minimal martingale measure prices, for each of the options. Note the difference in scales; for example, the width of the bounds for the 10-year option is less than for the other two options. Figures 5.4 - 5.6 show the good-deal bounds expressed as a ratio relative to the minimal martingale measure price.

Varying the good-deal bound B

The next step is to examine the sensitivity of the good-deal price bounds to the choice of the good-deal bound. The upper and lower good-deal price bounds for the three European put options as a function of the good-deal bound B are shown in Figures 5.7-5.9, for an initial stock price of 100. All the plots show that as we increase the good-deal bound B, the good-deal price bounds on the put option price widen. As we increase the good-deal bound B, we increase the range of values that the market price of regime change risk can take. This results in a wider range of prices. However, we notice in the plots where the market starts in regime 1 that the lower good-deal bound becomes constant. In this particular model with market parameters given by Table 5.1, the solution

Table 5.5: Black-Scholes Prices for European put options with strike price 100.

	Black-Scholes price		
Maturity	3-year	5-year	10-year
Market with regime 1 parameters	1.9631	1.3109	0.4422
Market with regime 2 parameters	17.5398	17.6373	14.3189

to the static optimization problem for the lower good-deal function is always $\bar{\eta}_{12}(t,x) = -1$ when starting in regime 1, no matter what the value of the good-deal bound B. Plugging $\bar{\eta}_{12}(t,x) = -1$ into the PIDE, we see immediately that the last term (5.1.1) vanishes. Thus the PIDE reduces to the classical Black-Scholes formula for a European put option in a non-regime-switching market with market parameters $r(1)$, $b(1)$ and $\sigma(1)$. These latter prices, and for completeness their counterparts for the non-regime-switching market with market parameters $r(2)$, $b(2)$ and $\sigma(2)$, are shown in Table 5.5.

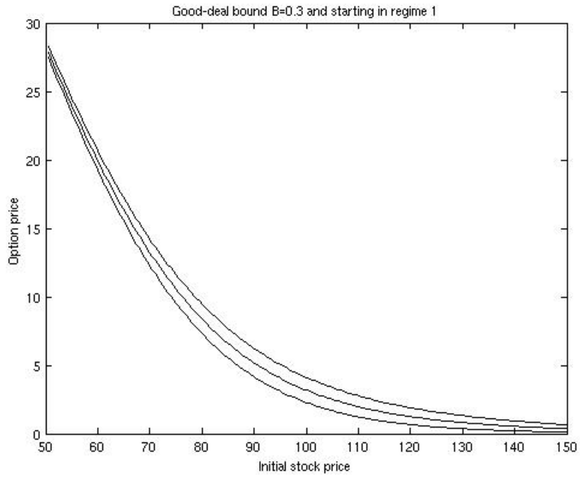
The graphs show how wide the pricing bounds can become as the good-deal bound B is increased. For example, the upper price bound for the 10-year put option increases from 2 at $B = 0.3$ to 8 at $B = 2$. As an aside, as these pricing bounds are all subsets of the no-arbitrage bounds, it illustrates why the latter are not considered useful in practice.

Conclusion

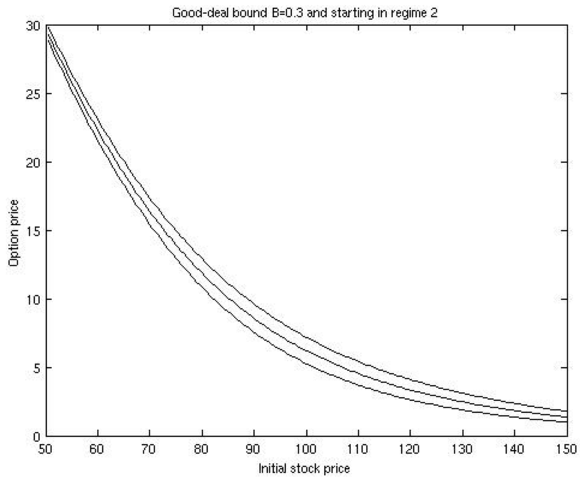
Adopting the good-deal bound approach, we have examined the resulting price bounds on three European put options of varying maturities. These bounds can be used as a guide to pricing maturity guarantees, by excluding those prices which imply a compensation which is too high for the risks undertaken. The overall message from the numerical example is that the pricing bounds are tight enough to be useful.

If the life insurance company has calculated a single price for a maturity guarantee then the good-deal pricing bounds can indicate whether or not a chosen price is reasonable. By choosing a Sharpe Ratio which reflects historical data to calculate the good-deal bound, it places the company's choice of a price in the context of what the market might reasonably choose as a price. It thus gives the company a reference point for pricing which is, to some degree, objective.

They can also be used directly for pricing. Once a life insurance company has picked the range of Sharpe Ratios that they consider reasonable, the corresponding good-deal pricing bounds can be calculated. The upper good-deal pricing bound can be viewed as the highest price at which the company will sell the maturity guarantee and the lower good-deal pricing bound as the lowest price.

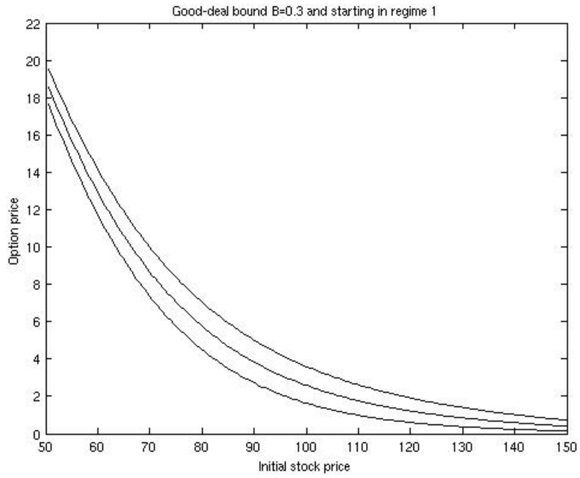


(a) Good-deal price bounds for a 3-year European put option starting in regime 1.

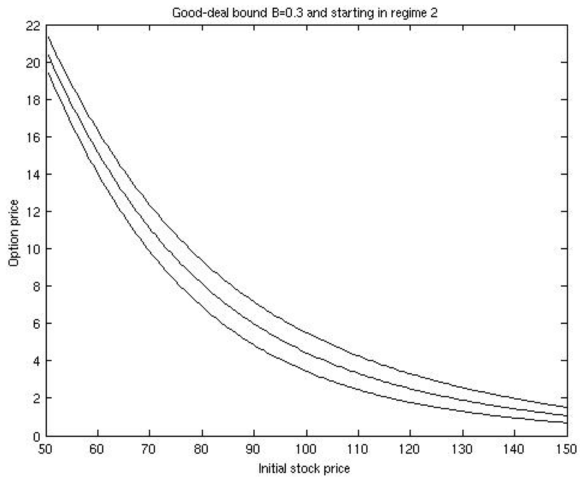


(b) Good-deal price bounds for a 3-year European put option starting in regime 2.

Figure 5.1: The upper and lower good-deal price bounds for a 3-year European put option with strike price 100 as a function of the initial stock price. The good-deal bound $B = 0.3$. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0. The plots show the upper and lower good-deal price bounds, with the minimal martingale measure price represented by the middle line.

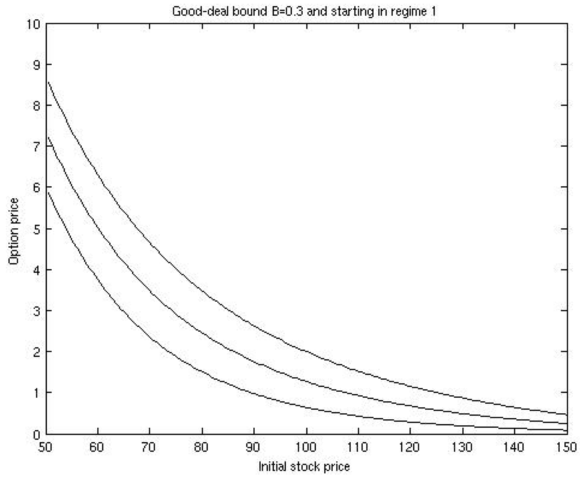


(a) Good-deal price bounds for a 5-year European put option starting in regime 1.

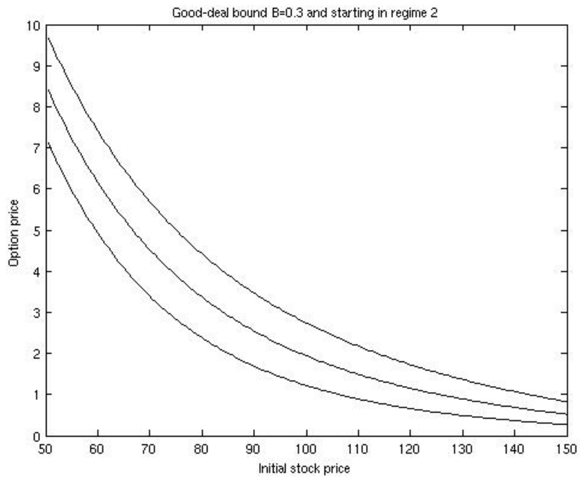


(b) Good-deal price bounds for a 5-year European put option starting in regime 2.

Figure 5.2: The upper and lower good-deal price bounds for a 5-year European put option with strike price 100 as a function of the initial stock price. The good-deal bound $B = 0.3$. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0. The plots show the upper and lower good-deal price bounds, with the minimal martingale measure price represented by the middle line.

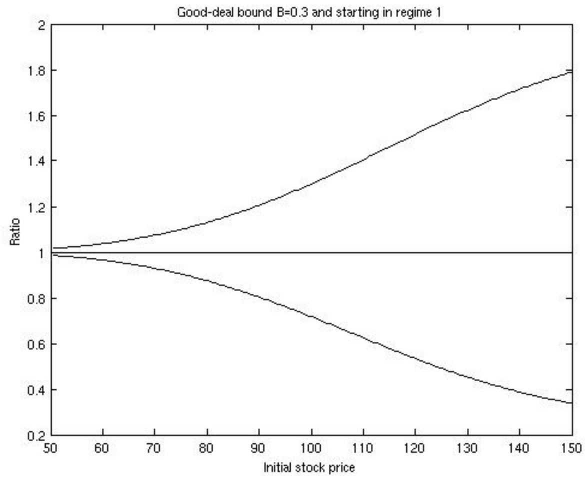


(a) Good-deal price bounds for a 10-year European put option starting in regime 1.

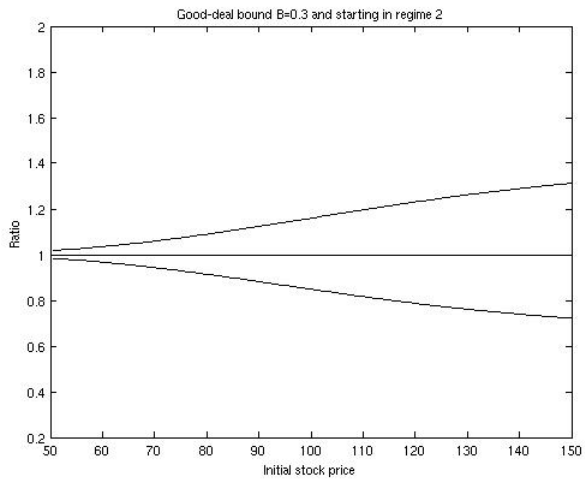


(b) Good-deal price bounds for a 10-year European put option starting in regime 2.

Figure 5.3: The upper and lower good-deal price bounds for a 10-year European put option with strike price 100 as a function of the initial stock price. The good-deal bound $B = 0.3$. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0. The plots show the upper and lower good-deal price bounds, with the minimal martingale measure price represented by the middle line.

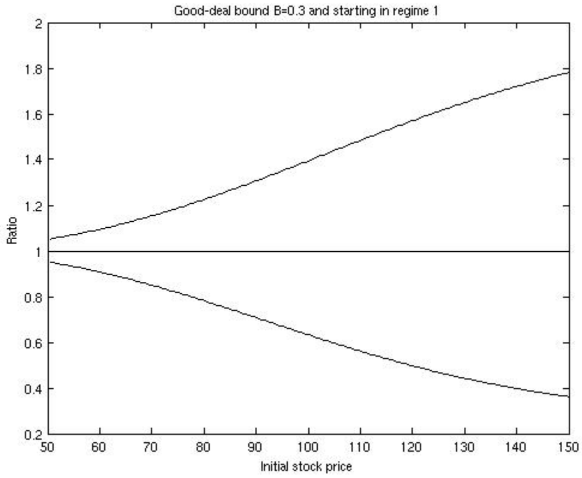


(a) Ratio of good-deal price bounds for a 3-year European put option starting in regime 1.

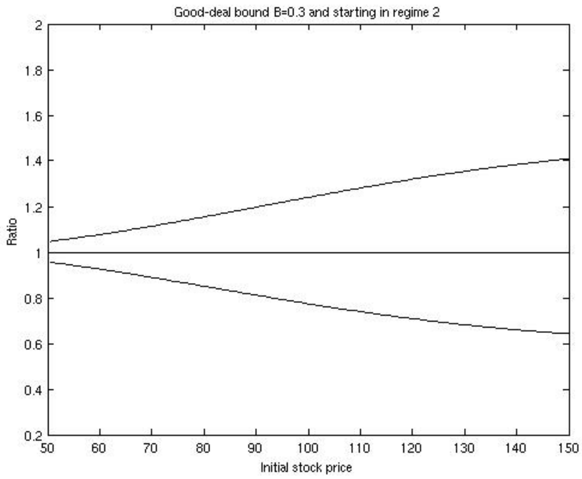


(b) Ratio of good-deal price bounds for a 3-year European put option starting in regime 2.

Figure 5.4: The ratio of the upper and lower good-deal price bounds to the minimal martingale measure price for a 3-year European put option with strike price 100 as a function of the initial stock price. The good-deal bound $B = 0.3$. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0.

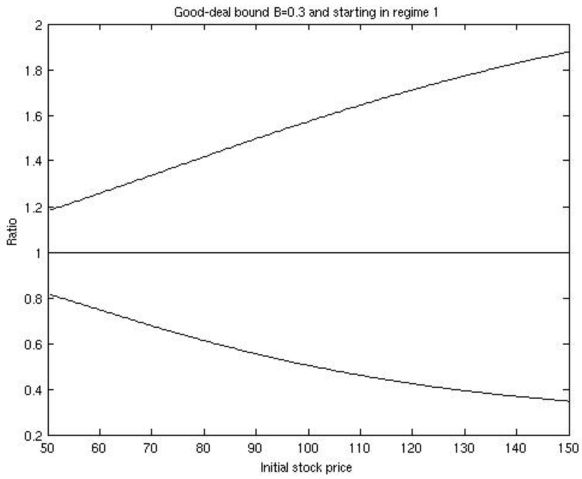


(a) Ratio of good-deal price bounds for a 5-year European put option starting in regime 1.

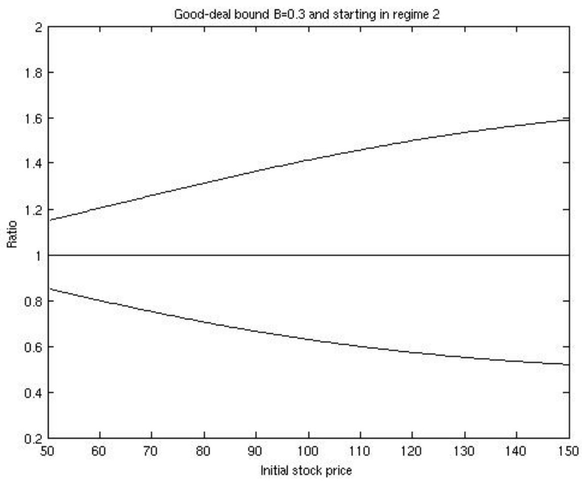


(b) Ratio of good-deal price bounds for a 5-year European put option starting in regime 2.

Figure 5.5: The ratio of the upper and lower good-deal price bounds to the minimal martingale measure price for a 5-year European put option with strike price 100 as a function of the initial stock price. The good-deal bound $B = 0.3$. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0.

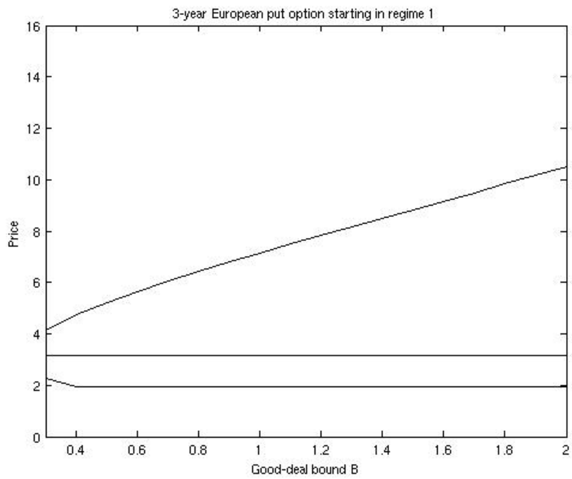


(a) Ratio of good-deal price bounds for a 10-year European put option starting in regime 1.

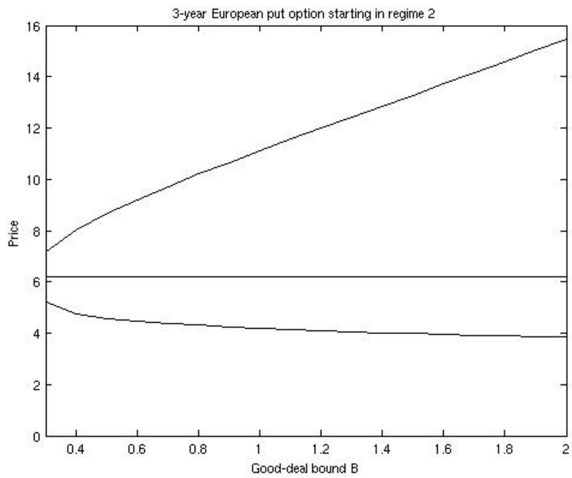


(b) Ratio of good-deal price bounds for a 10-year European put option starting in regime 2.

Figure 5.6: The ratio of the upper and lower good-deal price bounds to the minimal martingale measure price for a 10-year European put option with strike price 100 as a function of the initial stock price. The good-deal bound $B = 0.3$. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0.

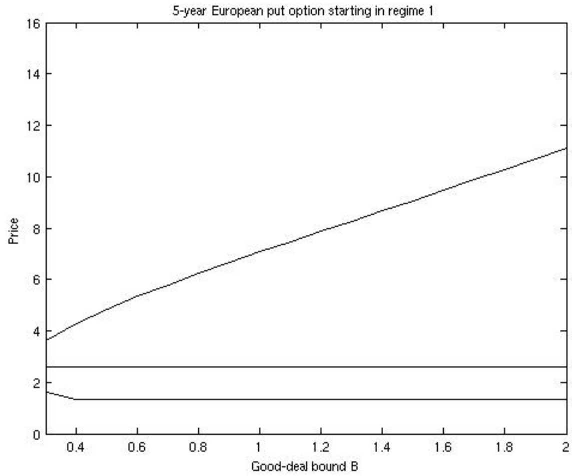


(a) Good-deal price bounds for a 3-year European put option starting in regime 1.

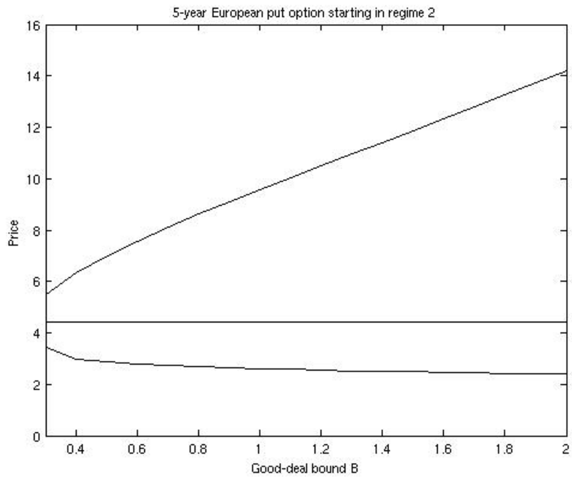


(b) Good-deal price bounds for a 3-year European put option starting in regime 2.

Figure 5.7: The upper and lower good-deal price bounds for a 3-year European put option with strike price 100 and initial stock price 100 as a function of the good-deal bound B. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0. On each plot, the minimal martingale measure price is the horizontal line in the middle.

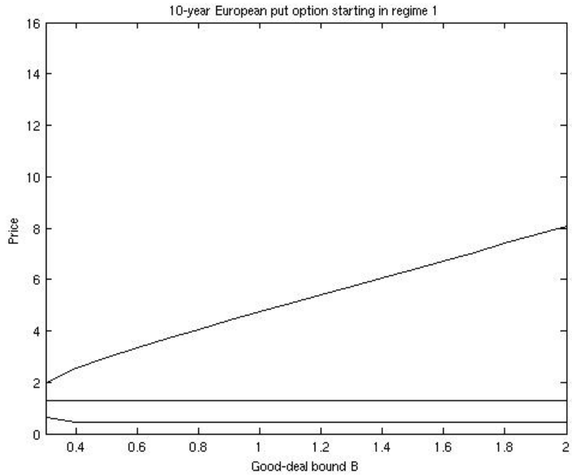


(a) Good-deal price bounds for a 5-year European put option starting in regime 1.

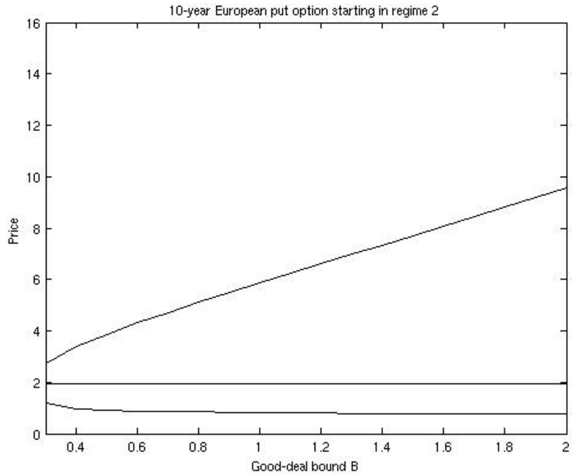


(b) Good-deal price bounds for a 5-year European put option starting in regime 2.

Figure 5.8: The upper and lower good-deal price bounds for a 5-year European put option with strike price 100 and initial stock price 100 as a function of the good-deal bound B. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0. On each plot, the minimal martingale measure price is the horizontal line in the middle.



(a) Good-deal price bounds for a 10-year European put option starting in regime 1.



(b) Good-deal price bounds for a 10-year European put option starting in regime 2.

Figure 5.9: The upper and lower good-deal price bounds for a 10-year European put option with strike price 100 and initial stock price 100 as a function of the good-deal bound B. The top plot assumes that the market is in regime 1 at time 0 and the bottom plot assumes that the market is in regime 2 at time 0. On each plot, the minimal martingale measure price is the horizontal line in the middle.

Chapter 6

Conclusion and outlook

We have applied the good-deal bound idea of Cochrane and Saá Requejo (2000) to a regime-switching lognormal market using the approach of Björk and Slinko (2006). The good-deal pricing bounds can be used as a guide to what constitutes a reasonable price for a maturity guarantee. We illustrated the approach with a numerical example, demonstrating that the resulting pricing bounds are tight enough to be useful in practice. It should be noted that, while the good-deal pricing bounds exclude extreme compensation for the risks undertaken, they do not exclude the possibility of extreme events. Thus events such as the recent financial crisis are not captured by the pricing bounds.

There are many interesting avenues for future research. The first is to examine how the many approaches to option pricing in the literature compare to one another. For example, how does the Esscher transform price compare to the minimal entropy martingale measure price, and how do these all compare to the good-deal pricing bounds?

Another line of research is to find approximations for the good-deal pricing bounds, instead of having to numerically solve the PIDE each time. This is currently been tackled by Björk and Slinko for jump-diffusion models (however, there does not appear to be a published paper yet), but nothing has been done yet for regime-switching models.

We have examined option pricing in the RSLN model and the flipside of this is *hedging*. What is a good-deal hedging strategy? To date, no work has appeared on this difficult problem on any incomplete market model but it is one of the challenges to be overcome in order to complete the good-deal approach.

Appendix A

Fully implicit finite difference method

Here we detail the method we used to solve numerically the PIDE (4.3.13)-(4.3.14) involved in the good-deal bound approach. There are various general techniques which can be implemented and we have chosen one - the fully implicit finite difference method - for its accuracy, rather than its speed. Our main reason for doing this is that no-one else has calculated the good-deal bounds in a RSLN model and thus we have no way of assessing the accuracy of the chosen method. Therefore, we prefer to err on the side of caution. However, there is no reason why a faster method could not be implemented; for example, Seydel (2009) details various methods for tackling PIDEs, although they must be modified to deal with the RSLN model.

We describe how we have implemented the fully implicit finite difference method in a general RSLN(D) model. Suppose that we have solved the static optimization problem, so that we know the values $\bar{\eta}_{ij}(t)$. The next step is to solve the PIDE

$$\begin{aligned}\frac{\partial V}{\partial t}(t,x,i)+r(i)x\frac{\partial V}{\partial x}(t,x,i)+\frac{1}{2}\sigma^2(i)x^2\frac{\partial^2 V}{\partial x^2}(t,x,i)\\ -r(i)V(t,x,i)+\sum_{j=1}^{D}g_{ij}(1+\bar{\eta}_{ij}(t,x))\left(V(t,x,j)-V(t,x,i)\right)=0.\end{aligned}$$

for each fixed triple (t,x,i). Note that $\bar{\eta}_{ij}(t)$ may depend on the sign of $V(t,x,j)-V(t,x,i)$. Fix (t,x), set

$$\lambda^{(i,j)}(t,x):=g_{ij}(1+\bar{\eta}_{ij}(t,x)),\qquad i\neq j, \tag{A.0.1}$$

and consider the solution $(f^{(i)})$ to the system of equations

$$\frac{\partial f^{(i)}}{\partial t}+r(i)x\frac{\partial f^{(i)}}{\partial x}+\frac{1}{2}\sigma^2(i)x^2\frac{\partial^2 f^{(i)}}{\partial x^2}-r(1)f^{(1)}+\sum_{j=1}^{D}\lambda^{(i,j)}(f^{(j)}-f^{(i)})=0, \tag{A.0.2}$$

for $i=1,\dots,D$. To solve this system, we construct a grid of the stock price against time. The grid is divided into N time points and M stock price points.

Denoting the terminal time by T and the maximum stock price by $S_{\max}$, this results in a grid size in the time direction of $\Delta t := T/N$ and in the stock price direction of $\Delta x := S_{\max}/M$. Thus the time $t := \nu\Delta t$ and the stock price $S := k\Delta x$ correspond to (ν, k) on the grid.

Denote the value of $f^{(i)}$ at point (ν, k) on the grid by $f^{(i)}_{\nu,k}$, and similarly denote the value of $\lambda^{(i,j)}$ at point (ν, k) by $\lambda^{(i,j)}_{\nu,k}$. The aim is to determine the time-0 values $(f^{(i)}_{0,k})$ for $k = 0, 1, \ldots, M$. These are the time-0 prices of the put option when the initial stock price is $S(0) = k\Delta x$ and the initial market regime is i. This is done by evaluating $(f^{(i)}_{\nu,k})$ at each point (ν, k) on the grid, working backwards from the terminal time values (corresponding to the points (N, k)) to time 0 values (corresponding to the points $(0, k)$).

To begin, we discretize the system of equations (A.0.2) using the first-order approximations

$$\begin{aligned}
\frac{\partial f^{(i)}_{\nu,k}}{\partial t} &\approx \frac{f^{(i)}_{\nu+1,k} - f^{(i)}_{\nu,k}}{\Delta t}, \\
\frac{\partial f^{(i)}_{\nu,k}}{\partial x} &\approx \frac{f^{(i)}_{\nu,k+1} - f^{(i)}_{\nu,k-1}}{2\Delta x}, \\
\frac{\partial^2 f^{(i)}_{\nu,k}}{\partial x^2} &\approx \frac{f^{(i)}_{\nu,k+1} - 2f^{(i)}_{\nu,k} + f^{(i)}_{\nu,k-1}}{(\Delta x)^2}.
\end{aligned}$$

Substituting the approximations into (A.0.2) and setting $x = k\Delta x$, we rearrange to find

$$\begin{aligned}
f^{(i)}_{\nu+1,k} = {} & \Delta t \left(\frac{1}{2} kr(i) - \frac{1}{2} k^2 \sigma^2(i) \right) f^{(i)}_{\nu,k-1} \\
& + \left(1 + \Delta t \left(k^2 \sigma^2(i) + r(i)\right)\right) f^{(i)}_{\nu,k} \\
& + \Delta t \left(-\frac{1}{2} kr(i) - \frac{1}{2} k^2 \sigma^2(i) \right) f^{(i)}_{\nu,k+1} \\
& - \sum_{j=1}^{D} \lambda^{(i,j)}_{\nu,k} \Delta t \left(f^{(j)}_{\nu,k} - f^{(i)}_{\nu,k} \right).
\end{aligned}$$

To write this more compactly, set

$$\begin{aligned}
a^{(i)}_k &:= \Delta t \left(\frac{1}{2} kr(i) - \frac{1}{2} k^2 \sigma^2(i) \right) \\
b^{(i)}_k &:= \left(1 + \Delta t \left(k^2 \sigma^2(i) + r(i)\right)\right) \\
c^{(i)}_k &:= \Delta t \left(-\frac{1}{2} kr(i) - \frac{1}{2} k^2 \sigma^2(i) \right) \\
d^{(i,j)}_{\nu,k} &:= -\lambda^{(i,j)}_{\nu,k} \Delta t.
\end{aligned} \tag{A.0.3}$$

so that we have the following system of $D(M+1)$ simultaneous equations

$$f^{(i)}_{\nu+1,k} = a^{(i)}_k f^{(i)}_{\nu,k-1} + b^{(i)}_k f^{(i)}_{\nu,k} + c^{(i)}_k f^{(i)}_{\nu,k+1} + \sum_{j=1}^{D} d^{(i,j)}_{\nu,k} \left(f^{(j)}_{\nu,k} - f^{(i)}_{\nu,k} \right), \tag{A.0.4}$$

for $i = 1, \ldots, D$, $\nu = 0, 1, \ldots, N$ and $k = 0, 1, \ldots, M$.

As we seek the value of a European put option, we use the following boundary conditions:

$$f_{N,k}^{(i)} = \max[K - S(T), 0], \qquad f_{\nu,0}^{(i)} = Ke^{-r(N-\nu)\Delta t}, \qquad f_{\nu,M}^{(i)} = 0. \qquad \text{(A.0.5)}$$

To solve (A.0.4) for each pair (ν, k), we work backwards from the final time $T = N\Delta t$ to time zero. From the boundary conditions, the values $(f_{N,k}^{(i)})$ are known for each $k = 0, 1, \ldots, M$. We set $\nu = N - 1$ in (A.0.4), solving to find $(f_{N-1,k}^{(i)})$ for each $k = 0, 1, \ldots, M$. However, note from the boundary conditions that we know both $(f_{N-1,0}^{(i)})$ and $(f_{N-1,M}^{(i)})$, so in fact we need only solve for $D(M-1)$ unknowns. Next we set $\nu = N - 2$ in (A.0.4) and use the solutions from the previous time step to find $(f_{N-2,k}^{(i)})$ for each $k = 1, 2, \ldots, M-1$. We continue working backwards like this through each of the time nodes until we reach time zero.

To see this more clearly, we write the system of $D(M-1)$ simultaneous equations at time step ν in matrix form. Define the matrices $(M-1)\times(M-1)$ matrices

$$A_i := \begin{pmatrix} b_1^{(i)} & c_1^{(i)} & 0 & 0 & \cdots & 0 \\ a_2^{(i)} & b_2^{(i)} & c_2^{(i)} & 0 & \cdots & 0 \\ 0 & a_3^{(i)} & b_3^{(i)} & c_3^{(i)} & & \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & a_{M-2}^{(i)} & b_{M-2}^{(i)} & c_{M-2}^{(i)} \\ 0 & & \cdots & 0 & a_{M-1}^{(i)} & b_{M-1}^{(i)} \end{pmatrix}$$

and

$$H_{ij}(\nu) := \begin{pmatrix} d_{\nu,1}^{(i,j)} & 0 & & \cdots & 0 \\ 0 & d_{\nu,2}^{(i,j)} & & & \\ & & & & \\ \vdots & & \ddots & & \vdots \\ & & & d_{\nu,M-2}^{(i,j)} & 0 \\ 0 & & \cdots & 0 & d_{\nu,M-1}^{(i,j)} \end{pmatrix}$$

for $i, j = 1, \ldots, D$. Define the $(M-1)$-column vectors

$$f_{\nu+1}^{\mathrm{adj}(i)} := \begin{pmatrix} f_{\nu+1,1}^{(i)} - a_1^{(i)} f_{\nu,0}^{(i)} \\ f_{\nu+1,2}^{(i)} \\ \vdots \\ f_{\nu+1,M-2}^{(i)} \\ f_{\nu+1,M-1}^{(i)} - c_{M-1}^{(i)} f_{\nu,M}^{(i)} \end{pmatrix}, \qquad f_{\nu}^{(i)} := \begin{pmatrix} f_{\nu,1}^{(i)} \\ \vdots \\ f_{\nu,M-1}^{(i)} \end{pmatrix}$$

and

$$f_{\nu}^{(i,j)} := \begin{pmatrix} f_{\nu,1}^{(j)} - f_{\nu,1}^{(i)} \\ \vdots \\ f_{\nu,M-1}^{(j)} - f_{\nu,M-1}^{(i)} \end{pmatrix}$$

Then, adjusting for the boundary conditions at $k = 0$ and $k = M$ (see (A.0.5)), the system of equations can be written in matrix form as

$$\begin{pmatrix} f_{\nu+1}^{\mathrm{adj}(1)} \\ \vdots \\ f_{\nu+1}^{\mathrm{adj}(D)} \end{pmatrix} = \begin{pmatrix} A_1 f_\nu^{(1)} \\ \vdots \\ A_D f_\nu^{(D)} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{D} H_{1j}(\nu) f_\nu^{(1,j)} \\ \vdots \\ \sum_{j=1}^{D} H_{Dj}(\nu) f_\nu^{(D,j)} \end{pmatrix}. \tag{A.0.6}$$

In most examples, $d_{\nu,k}^{(i,j)}$ (which form the diagonal entries of the matrix H_j) depends on the sign of $f_{\nu,k}^{(j)} - f_{\nu,k}^{(i)}$ (this is where the good-deal bound enters). This means that we cannot directly invert the right-hand side of (A.0.6). Instead, we must guess some initial values of $f_{\nu,k}^{(j)}$ and $f_{\nu,k}^{(i)}$ and then use an iterative technique to find the solution. Most mathematical software have inbuilt programs which can do this (for example, we used the inbuilt function *fsolve* in *MATLAB* to solve (A.0.6)).

For the results in Chapter 5, the calculations were done on a grid with values

$$\Delta t = 0.01, \quad \Delta S = 0.5, \quad S_{\min} = 0, \quad S_{\max} = 200,$$

where Δt is the grid step-size in the time direction (measured in years), ΔS is the grid step-size in the stock price direction and $[S_{\min}, S_{\max}]$ is the grid range in the stock price direction. The grid range in the time direction is $[0, T]$, where T is the maturity of the put option, in years.

Bibliography

A. Ang and G. Bakaert. Regime switches in interest rates. *Journal of Business and Economic Statistics*, 20:163–182, 2002.

E. Bayraktar and V. Young. Pricing options in incomplete equity markets via the instantaneous Sharpe ratio. *Annals of Finance*, 4:399–429, 2008.

A.E. Bernardo and O. Ledoit. Gain, loss, and asset-pricing. *Journal of Political Economy*, 108:144–172, 2000.

T. Björk. *Arbitrage Theory in Continuous Time.* Oxford University Press, Oxford, UK, 3rd edition, 2009.

T. Björk and I. Slinko. Towards a general theory of good-deal bounds. *Review of Finance*, 10:221–260, 2006.

P. P. Boyle and M. R. Hardy. Reserving for maturity guarantees: Two approaches. *Insurance: Mathematics and Economics*, 21:113–127, 1997.

J. Černý. Generalised Sharpe ratios and asset pricing in incomplete markets. *European Finance Review*, 7:191–233, 2003.

J. Cochrane and J. Saá Requejo. Beyond arbitrage: good-deal asset price bounds in incomplete markets. *Journal of Political Economy*, 108:79–119, 2000.

R.J. Elliott. Double martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 34:17–28, 1976.

H. Follmer and M. Schweizer. Hedging of contingent claims under incomplete information. In M. H. A. Davis and R. J. Elliott, editors, *Applied Stochastic Analysis*, pages 389–414. Gordon and Breach, London/New York, 1991. Volume 5.

M. Fritelli. The minimal entropy martingale measures and the valuation problem in incomplete markets. *Mathematical Finance*, 10:39–52, 2000.

H. U. Gerber and E.S.W. Shiu. Option pricing by Esscher transforms. *Transactions of Society of Actuaries*, 46:99–140, 1994.

S. F. Gray. Modeling the conditional distribution of interest rates as a regime-switching process. *Journal of Financial Economics*, 42(1):27–62, 1996.

J. D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, 57:357–384, 1989.

J. D. Hamilton and R. Susmel. Autoregressive conditional heteroskedasticity and changes in regime. *Journal of Econometrics*, 64:307–333, 1994.

L.P. Hansen and R. Jagannathan. Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99:225–262, 1991.

M. Hardy. *Investment Guarantees.* John Wiley, USA, 2003.

P.J. Hunt and J.E. Kennedy. *Financial derivativse in theory and practice.* John Wiley, USA, 2004.

J. Jacod. *Calcul Stochastique et Problemes de Martingales.* Springer, Berlin, Germany, 1979.

I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance.* Springer-Verlag, New York, USA, 1998.

F. Klaassen. Improving GARCH volatility forecasts with regime-switching GARCH. *Empirical Economics*, 27:363–394, 2002.

S. Klöppel and M. Schweizer. Dynamic utility-based good deal bounds. *Statistics and Decisions*, 25:285–309, 2007.

P. E. Protter. *Stochastic Integration and Differential Equations.* Springer-Verlag, New York, USA, 2nd edition, 2005.

L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales Volume 2.* Cambridge University Press, Cambridge, UK, 2nd edition, 2006.

R.U. Seydel. *Tools for computational finance.* Springer-Verlag, Berlin, Germany, 4th edition, 2009.