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MATCHING AND PORTFOLIO SELECTION: PART 2

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8. RECAPITULATION

8.1 This paper is concerned with the selection of investment portfolios to meet specified criteria which involve the liabilities of a long term investing institution such as a pension fund or a life office. In Part 1[5] I showed a certain relationship between, on the one hand, portfolios selected according to a criterion of pure matching to the liabilities and, on the other hand, portfolios selected according to a more general criterion of 'efficiency'. This connection points to a particular actuarial approach to the selection of portfolios, which is now further examined in Part 2.

8.2 Writing Part 2 separately presents an opportunity to re-state the main ideas. The next few paragraphs recapitulate the basic points with a view to reducing, within the subsequent discussion, the amount of cross-reference to Part 1, and to the three preceding papers[1] [2] [4] on which the study is based.

The nature of the model

8.3 We are dealing with a non-deterministic actuarial model which comprises the following elements:

(a) a portfolio of marketable assets with specified or estimated future cash flows, and aggregate present price P;
(b) a set of non-marketable liabilities with specified or estimated future cash flows;
(c) a statistical model for uncertain factors such as future rates of investment returns and inflation;
(d) the ultimate surplus from the portfolio when all the liability payments have been met, which can be regarded as a random variable;
(e) the mean of the ultimate surplus, E; and
(f) the variance of the ultimate surplus about its mean, V.

8.4 The portfolio price P is governed by the choice of assets which make up the aggregate portfolio, and by the prices of those assets. The mean and variance of the ultimate surplus, E and V, are governed by the cash flows resulting from the assets and the liabilities. Therefore E and V are liable to variation according to the choice of assets in the portfolio, according to variation in the liabilities, and according to changes in the actuarial assumptions with which the future cash flows are estimated. P is a present market value whilst E and V are values at the specified future date of accounting for ultimate surplus.

Criteria for portfolio selection

8.5 Given a particular set of liabilities and particular assumptions, the choice which is available regarding the asset portfolio gives a degree of control over P, E and V. If we attach conditions to these values we restrict the choice of portfolio. For example, one of the criteria for 'pure matching' is to minimise V subject to $E=0$. A criterion for an 'efficient portfolio' is to minimize V subject to specified P and E. It is interesting to ask: which criteria are appropriate for actuarial work?

8.6 The actuarial criterion put forward in Part 1 is specified in terms of E and another parameter called v. Before recalling the definition of v, it is useful first to present a diagrammatic view of the elementary case study.

The two-security case

8.7 The case study described by Wilkie(2) in sections 5 to 20 involved only two securities S_1 and S_2. The nominal holdings of each in a portfolio are denoted by x_1 and x_2 respectively. The equations quoted in § 2.6 (of Part 1) show how E, V and P are determined by the choice of x_1 and x_2.

8.8 Let us specify in our particular example that the mean ultimate surplus E is to be zero. The range of possibilities for x_1 and x_2 is reduced by this condition to one degree of freedom. Taking the price of security S_1 as 93 and that of S_2 as 95, the one-dimensional range of feasible portfolios (see § 3.14) is given by the following equations in this example:

$$x_1 = 1{\cdot}698 + {\cdot}32514\, v$$

$$x_2 = {\cdot}998 - {\cdot}31948\, v$$

8.9 Table 6 shows a few values of v, and the results for each.

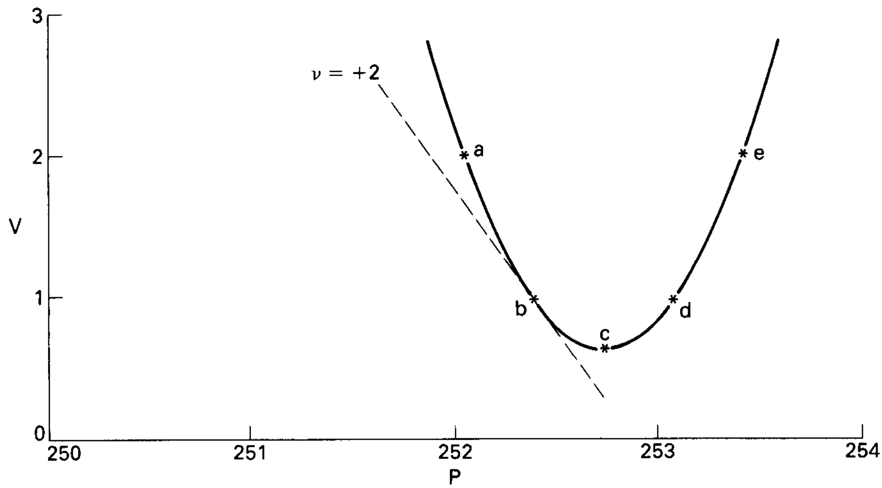
Table 6. *Prices $P_1=93$, $P_2=95$ and $E=0$.*

	Parameter	Portfolio		Characteristics	
Label	v	x_1	x_2	P	V
a	4	2·997	−·281	252·05	2·01
b	2	2·347	·359	252·39	·98
c	0	1·698	·998	252·74	·63
d	−2	1·048	1·638	253·08	·98
e	−4	·399	2·277	253·43	2·01

The V–P *graph*

8.10 We can plot the feasible portfolios as points in a plane whose axes give the measure of V and P. In our example, if we were to work out all feasible portfolios for all values of the parameter v, the result would be a parabolic line, as illustrated in Graph 1. The five portfolios shown in the above table are labelled in the diagram.

8.11 The parameter v traces out points on the parabolic line in this V–P graph. Each point marks a feasible portfolio, namely a portfolio which meets our



Graph 1. Securities S_1 and S_2.

condition that $E=0$. In §3.11 of Part 1, v was defined as the negative of the gradient of V with respect to P. Accordingly the value $v=0$ marks the apex of the parabola, namely the point of minimum V for given $E=0$. (This portfolio is the so-called unbiased match.) As v is increased from zero, the selected portfolio is shifted along the left arm of the parabola, in the direction of lower P and higher V. I have called v the 'degree of risk', because it measures the extent to which one wishes to accept a higher variance of ultimate surplus, or 'risk', in return for reduced price. (The word 'risk' is of course being used here in a long-term strategic sense.)

8.12 Negative values of v specify portfolios on the right arm of the parabola; these are 'inefficient portfolios' because they have both P and V higher than other portfolios (such as that at the apex) and are therefore unlikely to be of interest.

8.13 Another way of looking at v is to consider lines of the form $V+vP=k$. The number k can be varied to produce a range of such lines on the graph, all of which are parallel with slope $-v$. One such line is shown in Graph 1; it has slope -2 and it happens to touch the parabola at point b. If we consider the range of such parallel lines with $v=2$ it is apparent that the particular line shown on the graph is the one with the smallest value of k which intersects the parabola.

8.14 Generally, the portfolio selected by parameter v is the one which minimizes $V+vP$, for given E. This is another way of saying that v prescribes the desired balance between variance V and price P. If v is small or zero, this means that we are concerned to reduce the risk but we are not very concerned about the price of the portfolio. If v is large, relative to V, then we are trying to reduce the price of the portfolio and we are not too worried about the risk. In between the extreme choices for v will be a whole range of reasonable ones.

General formula

8.15 The formula in § 8.8 gives the asset holdings when $E=0$, and is a special case of the formula given in Part 1, § 3.14. The general solution for x_1 and x_2 in terms of E and v may be written thus:

$$x_1 = x_1^{\mathrm{o}} + Ey_1 + vz_1$$

$$x_2 = x_2^{\mathrm{o}} + Ey_2 + vz_2$$

(The values of x_1^{o}, x_2^{o}, y_1, y_2, z_1 and z_2 for the case study were summarized in § 6.2.)

8.16 This formula generalizes from 2 to n securities in the natural manner. Therefore if we use the vector $\boldsymbol{x} = (x_1, x_2, \ldots\ldots, x_n)$ to denote the nominal holdings in each security, the general formula for a portfolio in terms of E and v is (see § 5.3):

$$\boldsymbol{x} = \boldsymbol{x}^{\mathrm{o}} + E\,\boldsymbol{y} + v\boldsymbol{z}$$

8.17 The values of $\boldsymbol{x}^{\mathrm{o}}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are particular to each case in question, but they possess interesting general characteristics which were described in section 5 of Part 1. Briefly, $\boldsymbol{x}^{\mathrm{o}}$ is the unbiased match to the liabilities, $\boldsymbol{y}$ is to do with the expected return on the portfolio, and $\boldsymbol{z}$ is to do with the degree of risk.

8.18 This general formula provides a technique with which the actuary can specify the required mean ultimate surplus E and degree of risk v to determine the appropriate portfolio. The price P of that portfolio can be regarded as 'the value of the liabilities', but the liability value is of course dependent on the particular values assigned to the two parameters as well as all the actuarial assumptions.

8.19 The procedure which has been outlined so far leads to portfolios which may include negative holdings of assets, and this may be regarded as unrealistic. This brings us to the main point of Part 2, which is to consider the selection of optimum portfolios with prescribed values of E and v but with no negative asset holdings.

9. THE SELECTION OF POSITIVE PORTFOLIOS

9.1 The problem of finding portfolios without negative asset holdings (which I call positive portfolios) is rather similar to that of finding the positive match.[1] [4] The basic objective is to find out which assets occur in the selected positive portfolio. This is not a straightforward task; for example it would not be correct simply to work out the E, v portfolio to see which asset holdings were negative and then re-work the calculation as if such assets were not included in the model.

Case study

9.2 The problem may be explored by reference to the case study. To ensure that the example is non-trivial let us introduce a third security S_3, which yields cash flows of 10 at the end of the first year, 20 at the end of the second, and 100 at the end of the third. The cash flows resulting from all three securities, and from the liabilities, may be summarized thus:

Securities:	S_1:	(10, 100, 0)
	S_2:	(10, 10, 100)
	S_3:	(10, 20, 100)
Liabilities:		(100, 100, 100)

9.3 Suppose that the prices of the three securities, for unit amount, are:

$$P_1=93 \text{ and } P_2=95 \text{ (as before) and } P_3=103$$

Then the price P of the portfolio is:

$$P=93\ x_1+95\ x_2+103\ x_3$$

9.4 The variance V of ultimate surplus is given by the formula:

$$V=\boldsymbol{x}'\ \boldsymbol{V}\boldsymbol{x}-2\,\boldsymbol{x}'\,\boldsymbol{c}+V_L$$

where: $\boldsymbol{x}\ =(x_1,\ x_2,\ x_3)$

$\boldsymbol{V}$ is the matrix of covariances between the rolled-up returns on the three securities. For the given stochastic model:

$$\boldsymbol{V}=\begin{bmatrix} 1{\cdot}24176 & {\cdot}24366 & {\cdot}35456 \\ {\cdot}24366 & {\cdot}05556 & {\cdot}07646 \\ {\cdot}35456 & {\cdot}07646 & {\cdot}10736 \end{bmatrix}$$

$\boldsymbol{c}$ is the vector of covariances between the rolled-up return on the liabilities and the rolled-up returns on the three securities:

$$\boldsymbol{c}=(2{\cdot}43662 \quad {\cdot}55563 \quad {\cdot}76463)$$

and V_L is the variance of rolled-up return on the liabilities:

$$V_L=5{\cdot}55628$$

9.5 The mean ultimate surplus E is given by the formula:

$$E=\boldsymbol{x}'\ \boldsymbol{e}-E_L=x_1\ E_1+x_2\ E_2+x_3\ E_3-E_L,$$

where the E_i are the expected rolled-up returns on the three securities and the liabilities:

$$E=120{\cdot}881\ x_1+122{\cdot}781\ x_2+133{\cdot}681\ x_3-327{\cdot}810$$

Let us continue to specify $E=0$, restricting the feasible portfolios to two degrees of freedom.

9.6 A few feasible portfolios are shown in the following table, which includes the five portfolios of Table 6 as special cases with $x_3=0$.

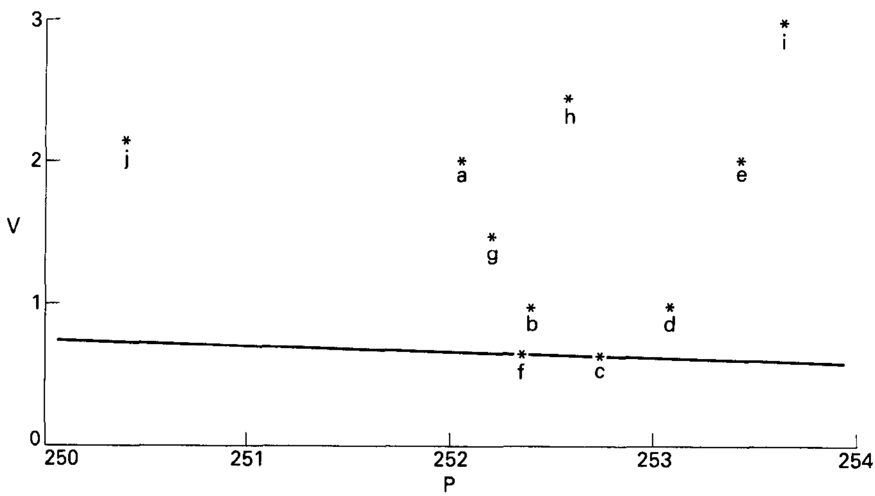
Table 7. *Prices* $P_1 = 93$, $P_2 = 95$, $P_3 = 103$ *and* $E = 0$

	Portfolio			Characteristics	
Label	x_1	x_2	x_3	P	V
a	2·997	−·281	·00	252·05	2·01
b	2·347	·359	·00	252·39	·98
c	1·698	·998	·00	252·74	·63
d	1·048	1·638	·00	253·08	·98
e	·399	2·277	·00	253·43	2·01
f	1·622	·00	·985	252·35	·65
g	2·712	·00	·00	252·20	1·47
h	·00	·00	2·452	252·57	2·45
i	·00	2·670	·00	253·64	2·99
j	·300	1·800	·500	250·40	2·16

V–P *graph*

9.7 If we were to plot all the feasible portfolios on the V–P graph, the result would be an area (because there is an infinity of portfolios with two degrees of freedom) bounded by a parabola. Graph 2 shows the boundary (which appears virtually straight over the chosen range) together with the ten portfolios of Table 7.

9.8 The boundary parabola marks the three-security portfolios with minimum variance for given P (and $E = 0$). Any such minimum variance portfolio $\boldsymbol{x} = (x_1, x_2, x_3)$ is determined by the formula given in Part 1, § 5.3 (subject to the



Graph 2. Securities S_1, S_2 and S_3.

necessary condition in each case that the covariance matrix V is non-singular). With $E=0$ the formula reduces to:

$$\boldsymbol{x}=\boldsymbol{x}^{o}+v\boldsymbol{z}$$

The parameter v therefore traces out the points along the boundary parabola, in a similar manner to the two-security case.

Combination of securities

9.9 The same general formula yields the minimum variance portfolio for any non-trivial combination of securities. Each combination gives rise to different values of $\boldsymbol{x}^{o}$ and $\boldsymbol{z}$ as shown in the following table. The first row gives the formula for the two securities S_1 and S_2, as shown in § 8.8, while the other rows enable the corresponding formulae for the other combinations of securities to be written down in a similar manner.

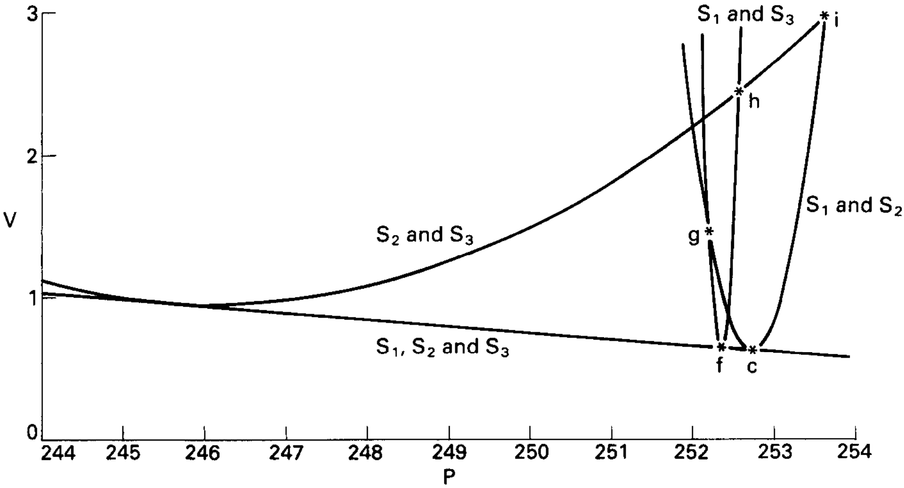
Table 8. *Alternative minimum variance portfolios*

Admitted securities	$\boldsymbol{x}^{o}$ x_1^o	x_2^o	x_3^o	$\boldsymbol{z}$ z_1	z_2	z_3
S_1, S_2	1·698	·998	·00	·325	−·320	·00
S_1, S_3	1·618	·00	·989	·100	·00	−·090
S_2, S_3	·00	−16·27	17·40	·00	−35·10	32·23
S_1, S_2, S_3	9	82	−81	−179·8	−1997·8	1997·5

9.10 It has to be said that the last line of Table 8 cannot be calculated exactly using the formula of Part 1 § 5.3 because the full covariance matrix V for all three securities (as shown in § 9.4) is singular and cannot be inverted. This is a special case arising from the fact that the number of securities equals the number of time steps and a special calculation procedure is needed to obtain the correct solution. As this particular aspect is incidental to the main point of the discussion, details are omitted.

9.11 The four parabolae for the four combinations of securities appear in the V–P plane as shown in Graph 3, which covers a wider range of P than before. Each parabola relating to a partial set of securities, such as S_1 and S_2, marks portfolios which are feasible and which therefore lie within the boundary parabola for S_1, S_2, and S_3. As the diagram shows, each of the three interior parabolae actually touches the boundary parabola tangentially. They must touch because as v traces the boundary parabola it attains values at which one of the x_i becomes zero. For example point f is attained when $v=·04104$ and $x_2=0$. At this point security S_2 drops out of the portfolio.

9.12 Other portfolios of particular interest are also marked on Graph 3 at points g, h and i. These are points at which pairs of securities drop out of the portfolio (see Table 7) and they appear at intersections of the two-security parabolae. It should be noted, in passing, that the intersection of two lines on the V–P graph does not necessarily imply a coincidence of portfolios. Each point on



Graph 3. Combinations of Securities S_1, S_2 and S_3.

the graph can represent more than one portfolio, each of which has the same V and P (and $E=0$).

Positive portfolios

9.13 Given that $E=0$ the region of positive portfolios, namely those without negative asset holdings, is obviously limited and finite. It would not be possible to obtain portfolios with very large or very small price P (relative to the range under consideration) without allowing a suitably large holding in one or two of the securities; this would require a corresponding negative holding in other securities in order to meet the condition of zero expected ultimate surplus.

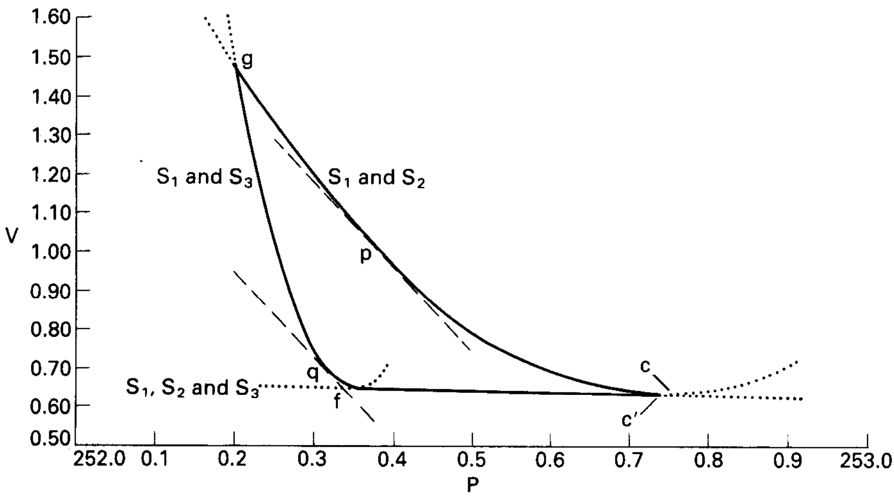
9.14 It is easy to identify the positive portfolios which are of interest, which must be arcs on the minimum variance parabolae shown in Graph 3. It can be verified that the four arcs are those identified as follows.

Table 9. *Positive minimum variance portfolios*

Admitted securities	Parabola arc
S_1, S_2	from g to i
S_1, S_3	from g to h
S_2, S_3	from h to i
S_1, S_2, S_3	from c' to f

where c' is the point adjacent to c at which the two parabolae touch. (c is the apex of the S_1, S_2 parabola and is therefore fractionally to the right of c'.)

9.15 As noted in § 8.12, we are not likely to be interested in the inefficient portfolios which appear on the right arm of any parabola. This immediately rules out the three arcs from c to i, from f to h, and from h to i, and enables us to focus attention on just a small area of the graph between the values $P=252$ and 253. The magnified picture appears in Graph 4. The positive portfolios of interest are those shown in solid lines.



Graph 4. Positive portfolios.

9.16 We can further rule out all portfolios with two securities S_1 and S_2 which lie on the arc from c' to g. None of these portfolios is optimal. For example if we consider a particular portfolio p on this arc, this portfolio optimizes $V+vP$ for portfolios in S_1 and S_2, but portfolio q on the arc from f to g achieves a lower value of $V+vP$ for the same parameter v. Any portfolio on the arc from c' to g can be bettered in this sense by an alternative portfolio to be found on the arcs from c' to f and from f to g.

9.17 It may be concluded that the efficient frontier of positive portfolios is represented along the very short arc from c to c' (securities S_1 and S_2), from c' to f (securities S_1, S_2 and S_3) and from f to g (securities S_1 and S_3). It is notable that the efficient frontier forms a continuous line from c (the positive unbiased match) to g (a portfolio consisting of the single security S_1). The switches from one parabola to another occur at tangent points, so the negative gradient v varies continuously along the efficient frontier from zero at c to a value of 10·93 at g.

9.18 Examples of portfolios along the efficient frontier are given in Table 10.

Table 10. *Efficient positive portfolios*

Parameter	Portfolio			Characteristics	
v	x_1	x_2	x_3	P	V
·00	1·698	·998	·00	252·74	·634
·04	1·711	·986	·00	252·73	·634
·041	1·630	·088	·897	252·39	·648
·05	1·623	·00	·984	252·35	·650
1·00	1·718	·00	·898	252·34	·656
5·00	2·118	·00	·537	252·28	·822
10·93	2·712	·00	·00	252·20	1·473

9.19 The pattern of these results is the same as before; as v is increased from the starting point of zero the price P of the resulting portfolio is decreased and the variance V of ultimate surplus is increased. The only differences are that the progression of P and V is not as steady as before, because of the switches from one combination of portfolios to another, and that the progression terminates.

Generalization

9.20 The significance of these observations is that they are equally true in any case with any number of securities. The V–P graph for n securities involves $2^n - n - 1$ minimum variance parabolae (each of which involves two or more securities), some touching and some crossing as shown in the three-security examples. The positive portfolios will always be finite arcs on these parabolae, and the efficient frontier will always form a continuous line of arcs with monotonic gradient v. Mathematically, the situation is just the same as in the classical portfolio selection problem.(6) We have merely replaced E in the classical problem by P.

9.21 Plainly if this type of analysis is to be used in practice, it will not be sensible to study graphs every time. Indeed it is not necessary to do so. The parameter v traces out the efficient frontier of positive portfolios, and all that is needed is a suitable quadratic programming algorithm to choose the right subset of securities for any given value of v. In my papers on matching(1) (2) I referred to a particularly suitable algorithm for finding the positive match portfolio. It turns out that a very similar algorithm is equally effective for determining efficient positive portfolios with any specified values of E and v. The mathematical details are not given here, but this particular algorithm was used to facilitate the production of all the further results quoted in this paper.

10. APPLICATION TO PENSION FUNDS

10.1 My paper on matching (1) showed how inflation could be brought into the model and it described examples of calculations made for a simplified model pension fund. The model was described in §§ 6.2 to 6.11 of that paper, and I shall examine the same model in the new context. I start by ignoring future contributions and using five year time steps (see § 6.12 of that earlier paper).

10.2 The matching portfolios for the pension fund model were effectively the same as those selected by parameter values $E=0$ and $v=0$. (A technical difference was mentioned in § 4.2 of Part 1 but it is of no consequence.) We can therefore examine a broader range of efficient portfolios with the earlier calculations of matching portfolios appearing as a special case.

10.3 The following table shows the result of keeping $E=0$ but increasing v until the selected portfolio reaches the terminal position of a single security. The starting value $v=0$ gives the same portfolio, with the same price of 1706, as the positive match.

Table 11. *Pension fund model ignoring future contributions—efficient frontier for $E=0$*

Parameter	Portfolio		Characteristics	
	Equity	Fixed interest		
v	% nominal	% nominal	P	V
0	70	30	1706	1,702
10	70	30	1705	1,708
100	64	36	1688	2,600
1000	35	65	1613	33,000
2000	21	79	1584	75,000
3000	10	90	1565	122,000
4000	0	100	1549	176,000
5000	0	100	1549	176,000

10.4 This particular model involves 12 different securities, in the form of six equity and six fixed interest investments which are sold or redeemed at different dates. The Appendix shows the detailed distribution of these assets in the various alternative portfolios, and the above summarizes the proportions by nominal amount (not market value) held in the two investment sectors. It will be seen that the effect of increasing v is to reduce the equity component and increase the fixed interest component until at the terminal position (for a value of v between 3000 and 4000) only a single fixed interest stock is held.

10.5 The earlier paper went on to look at the positive match for the same liabilities net of future contributions. In one of the two cases quoted, the price of the positive match was given as 1044 for an annual rate of contribution of 20 units in each five-year age group. Table 12 indicates the efficient frontier for these net liabilities.

10.6 The results in Table 12 show the features to be expected: the price of the portfolio is 1044 for $v=0$, and the price reduces while the variance increases with larger values of v. Clearly the factor of inflation is being dealt with adequately because the 'low risk' portfolios exhibit a balance of equity and fixed interest investments whilst the ultimate high risk portfolio is a single fixed interest security, whether future contributions are allowed for or not.

Table 12. *Pension fund model with future contributions—efficient frontier for $E=0$ (Case 1)*

Parameter	Portfolio		Characteristics	
	Equity	Fixed interest		
v	% nominal	% nominal	P	V
0	71	29	1044	6,552
10	70	30	1044	6,554
100	68	32	1041	6,700
1000	49	51	1012	22,000
2000	29	71	985	63,000
3000	11	89	966	109,000
4000	0	100	955	144,000

10.7 In view of § 5.15 of Part 1 it may be anticipated that the ultimate high risk portfolio, and therefore the nature of the various portfolios along the efficient frontier, will depend upon the ordering of the expected rates of return E_i/P_i for the different securities. To test this supposition, consider the alternative scenarios in Table 13, in which both current market values and future equity dividend growth are varied.

Table 13. *Alternative economic scenarios*

	Current market conditions		Actuarial assumption
	Fixed interest	Equity	
Case no.	redemption yield	dividend yield	Dividend growth
	% p.a.	% p.a.	% p.a.
1	11·0	4·8	4·5
2	8·0	4·8	4·5
3	11·0	6·8*	4·5
4	11·0	4·8	6·5
5	8·5	3·5*	4·5

* In all cases we assume here that the eventual sale of equities will be made on the basis of a 4·8% dividend yield. Only the current market conditions have been varied.

10.8 Case 1 is the scenario for the results shown above in Table 12, while the corresponding results for the altered market conditions of Case 2 are shown in Table 14.

10.9 In Case 2 the fixed interest securities are relatively unattractive, with a redemption yield of only 8·0%, and the ultimate high risk portfolio is all in equities. (The selection of 100% equities is reached at a relatively low value of v, but the initial range of durations to date of sale is narrowed down as v is increased beyond 1000.) An all-equity portfolio is high risk in this context because our model assumes fixed 3% annual increases on pensions—there is no allowance for extra increases if inflation is higher than expected. That is why the matching portfolio ($v=0$) involves a significant proportion of fixed interest stocks: 29% by nominal amount.

Table 14. *Pension fund model with future contributions—efficient frontier for E=0 (Case 2)*

Parameter	Portfolio: Equity	Portfolio: Fixed interest	Characteristics	
v	% nominal	% nominal	P	V
0	71	29	1139	6,552
100	74	26	1132	6,958
500	90	10	1092	19,000
1000	100	0	1080	27,000
2000	100	0	1071	39,000
3000	100	0	1065	55,000
4000	100	0	1060	73,000

10.10 Incidentally the composition of the matching portfolio is the same in Case 2 as in Case 1 because it is independent of current market values. Its price is higher, 1139 compared with 1044, because of the higher price of the fixed interest stocks in this scenario.

10.11 In Case 3 we look at an alternative situation involving a high dividend yield. The expected return on equities taking income and assumed future growth together is greater than the redemption yields on fixed interest stocks. The general pattern of the efficient frontier is therefore rather similar to that in Case 2.

Table 15. *Pension fund model with future contributions—efficient frontier for E=0 (Case 3)*

Parameter	Portfolio: Equity	Portfolio: Fixed interest	Characteristics	
v	% nominal	% nominal	P	V
0	71	29	826	6,552
100	74	26	818	6,937
400	94	6	779	18,000
1000	100	0	764	25,000
5000	100	0	750	65,000
10000	100	0	747	75,000

10.12 In Case 4 we have the same current market conditions as in Case 1 but the rate of future dividend growth, which is an actuarial assumption, has been increased.

Table 16. *Pension fund model with future contributions—efficient frontier for E=0 (Case 4)*

Parameter	Portfolio: Equity	Portfolio: Fixed interest	Characteristics	
v	% nominal	% nominal	P	V
0	63	37	887	5,253
100	64	36	878	5,796
500	70	30	834	19,000
1000	82	18	789	53,000
1500	98	2	752	98,000
2000	100	0	748	104,000

10.13 The result is again similar to Cases 2 and 3, because the high dividend growth assumption ensures that the expected return on equities exceeds that on fixed interest stocks. For a change the matching portfolio ($v=0$) now differs from the previous cases and it includes a larger proportion of fixed interest stocks. Changes in actuarial assumptions alter projected future cash flows, which variations in current market values do not. (We are of course free to amend the actuarial assumptions in the light of current market conditions.) In this case the higher dividend growth assumption presumably has the effect of making the equities less well matched to the liabilities because of their lengthened mean term.

10.14 In Case 5 the redemption yield on fixed interest stocks (8·5%) is only marginally greater than the expected return on equities ($1{\cdot}035 \times 1{\cdot}045 = 1{\cdot}082$). All the expected returns are below the 9% valuation rate of interest.

Table 17. *Pension fund model with future contributions—efficient frontier for $E=0$ (Case 5)*

Parameter	Portfolio		Characteristics	
	Equity	Fixed interest		
v	% nominal	% nominal	P	V
0	71	29	1396	6,552
100	65	35	1379	7,447
1000	19	81	1242	76,000
10^4	0	100	1187	$\cdot 2 \times 10^6$
10^5	0	100	1163	$1{\cdot}1 \times 10^6$
10^6	0	100	1159	$1{\cdot}6 \times 10^6$

10.15 As could be expected, a fixed interest stock figures in the ultimate high risk portfolio. This end of the efficient frontier is not reached until a very high value of v is specified, because of the closeness between the expected yields on the various securities. It will also be seen from the Appendix that the effect of increasing v is to move the selected portfolio into the securities which are held for the shorter durations. The explanation for this feature is that the expected return on a security includes interest at the valuation rate on reinvestment of the projected future proceeds of income and capital. In this particular case the advantage lies with the shortest fixed interest stock, because of the 9% assumed rate of return after redemption of the investment.

10.16 The final example (Table 18) returns to the first example discussed in §§ 10.1 to 10.4, with the introduction for the first time of a positive expected surplus. The details of the selected portfolios are given in the Appendix. They are fairly similar to those for $E=0$ (see Table 11 and Appendix), but with a greater emphasis on the fixed interest stocks and the assets of longer term.

Table 18. *Pension fund model ignoring contributions—efficient frontier for $E = 1000$*

Parameter	Portfolio		Characteristics	
	Equity	Fixed interest		
v	% nominal	% nominal	P	V
0	64	36	1771	3,410
10	64	36	1769	3,420
100	59	41	1754	4,200
1000	35	65	1694	28,000
2000	23	77	1666	70,000
3000	12	88	1648	116,000
4000	2	98	1630	178,000

11. CONCLUSION

11.1 In this paper I have sought to reason that the expected surplus E and the degree of risk v are appropriate parameters which an actuary may use to determine strategic portfolio structures relative to long term liabilities. In any given situation there is always an efficient frontier of positive portfolios; these are the portfolios with no negative asset holdings which are optimal in the sense that any other such portfolio with the same price P and expected surplus E will give a larger variance V of ultimate surplus. The concept is very similar to that of classical portfolio theory except that the long term liabilities are included and the price P of the portfolio is substituted for E in the trade-off against V.

11.2 The efficient frontier is a one-dimensional range of portfolios which can be traced out by the parameter v. At one end of the range $v = 0$, the variance is minimized and the portfolio is the positive unbiased match as defined in my earlier papers.[(1)(4)] At the other extreme the ultimate high risk portfolio consists of a single security, the one with the highest expected return. From one end of the efficient frontier to the other, the parameter v can be used to trace all the efficient portfolios. As v is increased the composition of the portfolio changes steadily from the matching to the ultimate high risk portfolio, the price of the portfolio reduces and the variance of ultimate surplus increases. There are no sudden jumps or discontinuities in any of these features.

11.3 The composition of the matching portfolio is independent of current market conditions, but that of the ultimate high risk portfolio is not. In consequence, the efficient frontier is liable to short term fluctuations with changing market conditions, although the degree of this effect on any particular choice of portfolio will depend on the extent of departure from the matching position.

11.4 The efficient frontier is also dependent on the actuarial assumptions. It might therefore be difficult to justify and explain the technique for practical applications, although such considerations have not inhibited actuaries in the past! It is clear that analysis of sensitivity to alternative assumptions would be necessary before attempting to draw conclusions from the numbers.

11.5 The theory is not affected by the complexity of the liability model, and it is apparent that any degree of complexity can be represented and successfully dealt with providing that the future cash flows can be estimated.

11.6 The calculations reported in this paper have all been based on an analytical treatment of the stochastic model of interest and inflation. Appendix C to my earlier paper[4] showed how to derive exact figures for the covariance-type matrix. However, analytical methods are not able to deal with the more realistic stochastic models for investment returns and inflation which actuaries might wish to use in practice. In particular the long term stochastic model which has been developed by Wilkie[7] is too complicated to be applied in the analytical way.

11.7 There is however another way forward, and that is by use of simulation methods. The portfolio selection technique requires for its basic data the expected returns on the liabilities and each of the available securities, and the matrix of covariances between all these accumulated amounts. If we used a stochastic model as a basis for numerous simulations of the asset and liability cash flows, we could arrive at estimates of all the required means and covariances. The estimates could be made as accurate as we wished by doing enough simulations, subject to the constraints of computing time. We could even build in the effects of financial options and guarantees in the simulations of asset and liability cash flows, and these effects would be represented in the mean and covariance estimates. Some work on this has been done, and the indications are that realistic practical applications can be put in hand in this way.

11.8 I conclude by expressing my grateful thanks to all those whose own efforts helped me to produce this paper: especially to David Wilkie, to Mike James and other colleagues who produced and verified numbers and graphs from the computer, and to Margaret Payne who produced the typescript.

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APPENDIX—EFFICIENT FRONTIERS FOR THE PENSION FUND MODEL

Table 11 (§ 10.3) shows the sector distribution of selected efficient portfolios. The nominal holdings in each of the 12 securities are given below for each of the selected portfolios.

Table 11 *supplement*

Asset redemption dates		Parameter 0	10	100	1000	2000	3000	4000
Equity	1–5	176	173	137	0	0	0	0
	6–10	171	169	144	8	0	0	0
	11–15	314	312	283	133	24	0	0
	16–20	420	418	374	189	7	0	0
	21–25	127	127	164	0	0	0	0
	26–30	0	0	0	241	315	160	0
Fixed interest	3	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0
	13	0	0	0	0	0	0	0
	18	0	0	0	0	0	0	0
	23	512	520	68	0	0	0	0
	28	0	0	540	1082	1285	1459	1609

Similar information is shown below for Table 17 (§ 10.14) and Table 18 (§ 10.16).

Table 17 *supplement*

Asset redemption dates		Parameter 0	100	1000	10^4	10^5	10^6
Equity	1–5	0	0	0	0	0	0
	6–10	21	7	0	0	0	0
	11–15	200	185	0	0	0	0
	16–20	382	325	0	0	0	0
	21–25	142	168	0	0	0	0
	26–30	0	0	189	0	0	0
Fixed interest	3	0	0	0	0	762	947
	8	0	0	0	296	189	0
	13	0	0	283	677	0	0
	18	0	92	523	0	0	0
	23	309	270	0	0	0	0
	28	0	0	0	0	0	0

Table 18 *supplement*

Asset redemption dates		Parameter 0	10	100	1000	2000	3000	4000
Equity	1–5	129	125	91	0	0	0	0
	6–10	135	132	109	0	0	0	0
	11–15	283	280	255	123	15	0	0
	16–20	394	389	360	188	6	0	0
	21–25	211	215	230	0	0	0	0
	26–30	0	0	0	303	373	210	29
Fixed interest	3	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0
	13	0	0	0	0	0	0	0
	18	0	0	0	0	0	0	0
	23	216	166	0	0	0	0	0
	28	425	485	736	1120	1321	1493	1663